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Forced in-plane vibration of a thick ring on a unilateral elastic foundation

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ABSTRACT

Most existing studies of a deformable ring on elastic foundation rely on the assumption of a linear foundation. These assumptions are insufficient in cases where the foundation may have a unilateral stiffness that vanishes in compression or tension such as in non-pneumatic tires and bushing bearings. This paper analyzes the in-plane dynamics of such a thick ring on a unilateral elastic foundation, specifically, on a two-parameter unilateral elastic foundation, where the stiffness of the foundation is treated as linear in the circumferential direction but unilateral (i.e. collapsible or tensionless) in the radial direction. The thick ring is modeled as an orthotropic and extensible circular Timoshenko beam. An arbitrarily distributed time-varying in-plane force is considered as the excitation. The Equations of Motion are explicitly derived and a solution method is proposed that uses an implicit Newmark scheme for the time domain solution and an iterative compensation approach to determine the unilateral zone of the foundation at each time step. The dynamic axle force transmission is also analyzed. Illustrative forced vibration responses obtained from the proposed model and solution method are compared with those obtained from a finite element model.

1. Introduction

This paper deals with the analysis of the in-plane vibration of non-pneumatic tires [1] and other similar structures using a deformable ring on elastic foundation (REF) model, where the foundation may have a non-linear or unilateral stiffness [2,3]. The in-plane vibration of a REF [4,5] has been intensively studied due to its broad and important applications such as in tires [6–8], wheels and gears [9,10]. Most of the existing studies on REFs assume a distributed elastic foundation, whose stiffness is uniformly constant around the ring. These REF models with linear foundation assumptions can be solved analytically [5,11]. The ring resting on this linear elastic foundation has been treated using different beam models and theories. The simplest model proposed for the ring is a tensioned string that has direct tensile strain but no bending stiffness [12]. More practically, a thin ring is modeled using Euler–Bernoulli beam theory, where both the tensile and the bending stiffness are taken into account by assuming that plane cross-sections remain plane and are always normal to the neural axis of the

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Abbreviations: REF, ring on elastic foundation; EOMs, Equations of Motion; FEA, Finite Element Analysis
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Nomenclature

R	radius of ring centroid
h	thickness of ring in radial direction
b	model width in perpendicular direction of ring plane
K_r, K_θ	radial and circumferential stiffness per radian of foundation
E_r, E_θ, G	radial, circumferential and shear modulus of the ring
A, I	area and area moment of inertial of ring cross-section
$C_{Er}, C_{E\theta}$	foundation viscous damping per radian in radial and circumferential direction
ρ	mass density of ring
u_r, u_θ	radial and circumferential displacement
ϕ	rotation angle of ring cross-section
r, θ	radial and circumferential coordinate
t	time
n, N	mode number and cut-off mode number
$\epsilon_{rr}, \epsilon_{\theta\theta}, \gamma_{r\theta}$	radial, circumferential and shear strain
$\sigma_{rr}, \sigma_{\theta\theta}, \tau_{r\theta}$	radial, circumferential and shear stress
ν	Poisson's ratio
U_1, U_2, T, W	strain energy of ring, strain energy of foundation, kinetic energy of ring and work done by external force
q_r, q_θ	distributed external force per unit area
F_r, F_θ	distributed external force per radian
Q, H	Fourier coefficients of external force and compensation force
a, b, c, p	time coefficients for radial, circumferential displacement, cross-section rotation and external force
$\mathbf{M}, \mathbf{C}, \mathbf{K}$	mass, damping and stiffness matrices in Equation of Motions
$\mathbf{A}(t), \mathbf{V}(t), \mathbf{X}(t), \mathbf{\Gamma}(t)$	acceleration, velocity, displacement and forcing vectors in Equation of Motions
α, β	weight coefficients in Newmark method
σ	distribution factor of Gaussian function
k_r, k_θ	non-vanished spring element stiffness in FEA model

ring after deformation [8,11]. Thick rings are often modeled using Timoshenko beam theory, which takes the shear deformation into account by assuming the normal of a plane cross-section is subjected to rotation in addition to the bending effects [13]. The effect of the extensibility of the ring has also been addressed by combining both thin and thick ring models with linear and uniform elastic foundations [14,4].

Allaei et al. [15], Wu and Parker [10] extended the studies to linear but non-uniform elastic foundations. Allaei et al. [15] studied the natural frequencies and mode shapes of rings supported by a number of radial spring elements attached at arbitrary locations. Wu and Parker [10] studied the free vibration of rings on a general elastic foundation, whose stiffness distribution can be different circumferentially, and gave the closed-form expression for natural frequencies and vibration modes. In this case, however, the circumferential distribution of the stiffness is fixed and known. The application of this non-uniform elastic foundation includes tires with non-uniformity and planetary gears where tooth meshes for the ring and the planets are not equally spaced.

The structure of the non-pneumatic tire invented by co-authors [1] consists of a deformable shear ring supported by collapsible spokes which buckle and lose stiffness when compressed [16]. The stiffness given by this kind of unilateral foundation typically vanishes locally in the compressed region. This unilateral REF model is also applicable to the bushing bearings, whose external sleeve can lose contact with the internal sleeve which can then be modeled as a ring on tensionless foundation. This deformation-dependent stiffness makes the dynamics of the REF model nonlinear and thus more difficult to solve. This nonlinear dynamic model can be solved via numerical methods such as Finite Element Analysis (FEA), but a parametric REF model that does not rely on discretization and mesh generation is more desirable since it can facilitate rapid design space exploration, and offer broadly useful physical insights into the major effects. However, the existing body of literature on analysis of parametric REF models with unilateral foundations is rather limited. Celep [17] studied the forced response of a thin and inextensible circular ring on tensionless two-parameter foundation under a time varying in-plane load. Direct numerical integration was used to solve the nonlinear differential equations. The circumferential displacement of the ring was obtained from the inextensible ring assumption. This approach cannot be adopted for a more general extensible Timoshenko ring. Gasmi et al. [2] studied a Timoshenko ring resting on a collapsible foundation for the analysis of non-pneumatic tires, but their work was only limited to the static problem and could not be straightforwardly extended to the dynamic case.

The present paper deals with the in-plane vibration of a deformable thick ring resting on unilateral foundation. In our companion paper [3], which focused on the static deformation of a thick ring on unilateral foundation, we proposed an iterative compensation scheme to analyze the unilateral foundation problem. The approach was built from the analytical solution of the linear foundation case by first computing the excessive force that would not be there with a unilateral foundation. Then, a compensation force is applied to the linear foundation model to counter-act this excessive force thereby setting up a simple algebraic iteration scheme to solve for the final deformation for the unilateral foundation case. In this paper, the iterative compensation approach is extended to solve the dynamic problem where the forced vibration of a thick ring on a unilateral elastic foundation is investigated. The ring is modeled as an orthotropic circular Timoshenko beam and the foundation is assumed to be a two-parameter elastic foundation, which has both radial and circumferential stiffness but the one in the radial direction is unilateral. Linear viscous damping is incorporated into the Equations of Motions (EOMs)