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Unique determination of a single crack in a uniform simply supported beam in bending vibration

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ABSTRACT

In this paper we consider one of the basic inverse problems in damage detection based on natural frequency data, namely the identification of a single open crack in a uniform simply supported beam from measurement of the first and the second natural frequency. It is commonly accepted in the literature that the knowledge of this set of spectral data allows for the unique determination of the severity and the position (up to symmetry) of the damage. However, in spite of the fact that many numerical evidences are in support of this property, the result is rigorously proved only when the severity of the crack is small. In this paper we definitely show, by means of an original constructive method, that the above result holds true for any level of crack severity.

1. Introduction

Dynamic methods based on natural frequency measurements are widely used as powerful tool for crack detection in beam structures. Resonant frequencies are often chosen as input data since they are easy to measure in experiments, and are subject to errors less than those affecting other dynamic data, such as, for instance, mode shape components. In addition, in case of ideal undamped systems, it is somewhat more easy to extract information on possible occurrence of damage from natural frequency measurements than from other dynamical parameters, particularly for concentrated damages, such as cracks or notches in beams [1].

After the appearance of the pioneering paper by Adams et al. [2], an extensive literature is nowadays available on the identification of defects in beams structures by frequency measurements; see, among others, the contributions by [3, Chapter 15] and [4] for an introduction to the topic, and [5–7] for recent advances on multiple crack identification in beams and in frames, respectively. In spite of this, several basic, fundamental diagnostic problems are still open. Their study is useful both for the application of dynamic techniques in practice and for the definition of a comprehensive theory of damage detection in structures.

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One of these basic problems is considered here. Specifically, we deal with the inverse problem of determining a single open crack in a uniform simply supported beam from the first two natural bending frequencies. The damage is modelled as a massless linearly elastic rotational spring located at the damaged cross-section [8]. The main contribution of the present research is the rigorous unique determination of the crack position and severity, by means of a constructive algorithm, without any a priori assumption on the smallness of the damage. It should be recalled, in fact, that a well-established theory is available for our problem in case of small crack [9–11]. The smallness of the damage allows us to linearize the inverse problem in a neighborhood of the undamaged configuration, see also [12] for a theory which includes second-order terms in the eigenvalues expansion with respect to the crack severity. Then, taking advantage of the closed-form expression of the eigenpairs of the uniform undamaged beam, it is possible to obtain exact solutions of the linearized inverse problem, with closed form expressions both the position and the severity of the crack in terms of the data. In particular, it turns out that the first two natural frequencies determine uniquely the small crack, up to a symmetrical position with respect to the mid-point of the beam.

When the damage is not small, the linearization is no longer allowed and one has to deal with the full nonlinear crack identification problem. There are strong motivations in support of the extension of the theory to not necessarily small cracks. Firstly, it is not easy to rigorously state when a crack can be considered small. Secondly, the linearized theories by Narkis [9] and Morassi [10] show some limitations when the damage is located near a point of vanishing sensitivity for a vibration mode. Third, it is obviously desirable to have a unifying theory of the diagnostic problem capable to include any severity of the damage.

In this paper we prove, by means of a constructive argument, that the measurement of the first and the second natural frequency of the cracked beam is sufficient for the unique determination of the crack (up to a symmetric position) for any level of severity of the damage. However, unlike the corresponding linearized problem, no closed-form expressions of the damage parameters in terms of the frequency data are available.

Our method differs from that recently presented in [13] for the analysis of the analogous crack identification problem in a uniform beam under longitudinal vibration. The analysis developed in [13] was essentially based on the Frequency Equation Method, that is, a careful study of the solutions of the nonlinear system formed by the frequency equation – which is available in closed form, since the system has constant coefficients – written for the two selected resonant frequencies in terms of the position and severity of the crack. The analysis of the corresponding nonlinear system for the cracked beam in bending turns out to be significantly more difficult, due to the simultaneous presence of harmonic ($\cos$, $\sin$) and hyperbolic ($\cosh$, $\sinh$) functions. In particular, it has proved difficult to find a complete characterization of the set of admissible natural frequency data for which the existence and uniqueness of the solution to the inverse problem are ensured. In view of these difficulties, we had to follow a different approach that, at the end, resulted in an original constructive algorithm for the identification of the damage parameters.

We now describe the main steps of our approach. The proof of the result is based on three main steps. In a first step, we transform the eigenvalue problem of the cracked beam in an equivalent eigenvalue problem for a simply-supported beam carrying a point mass $m=\frac{1}{K}$ at the cracked cross-section of abscissa s, where K is the stiffness of the linearly elastic rotational spring modelling the crack (see Proposition 3.1 in Section 3). Hence, the crack detection problem is transformed into the equivalent problem of determining the location s and the magnitude m of the point mass from the first two natural frequencies of the beam. In the second step, we study the $\lambda-m$ and $\lambda-s$ curves, that is the functions $\lambda=\lambda(s,\cdot)$ and $\lambda=\lambda(\cdot,m)$, for fixed s and fixed m, respectively, where λ is the first and the second eigenvalue (Section 4). This analysis is based on the determination of the explicit expression of the eigenvalue first partial derivatives with respect to the parameter s and m (Proposition 4.1), and on specific properties of the $\lambda-m$ and $\lambda-s$ curves of the cracked beam (Propositions 4.2 and 4.3). These properties are used in the last step to prove the main result (Section 5). The proof is constructive and leads to an identification method, called λ-Curves Method, which is alternative, although equivalent, to the Frequency Equation Method. It should be noted that the proof relies on a sharp lower bound for the second eigenvalue of the cracked beam (Proposition 3.5). Such a bound plays an important role in our treatment and follows from a careful analysis of the frequency equation of the cracked beam.

Once the existence and uniqueness of the solution to the inverse problem are proved, a specific crack identification problem can be addressed by using either the Frequency Equation Method or the λ-Curves Method. Our experience shows that, although the use of the λ-Curves Method was necessary in the proof of the main theorem, the numerical implementation of the Frequency Equation Method is less onerous. An extensive series of numerical simulations with various positions and severities of the crack support the theory (Section 6).

Finally, for the sake of completeness, we recall that several contributions on the crack identification problem considered in this paper are available. Apart from variational techniques, see, for example, [14–21], the approach generally adopted consists in solving numerically the nonlinear system of the frequency equations written for the first two natural frequencies. We refer, among others, to the studies carried out in [22–26]. All the known results support the conjecture that the inverse problem has positive answer. However, at the best of our knowledge, a rigorous proof of this general property was not available, as the conclusions of the above studies were drawn either on the basis of numerical analysis of specific cases, or on the study of particular experimental situation.