Journal of Sound and Vibration 366 (2016) 149–172

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

Reconstructing blockages in a symmetric duct *via* quasi-isospectral horn operators

Antonio Bilotta [a], Antonino Morassi [b,1], Emilio Turco [c,*]

[a] *Università della Calabria, Dipartimento di Ingegneria Informatica, Modellistica, Elettronica e Sistemistica - DIMES, Via P. Bucci, 87036 Rende, CS, Italy*
[b] *Università degli Studi di Udine, Dipartimento Politecnico di Ingegneria e Architettura, via Cotonificio 114, 33100 Udine, UD, Italy*
[c] *Università degli Studi di Sassari, Dipartimento di Architettura, Design e Urbanistica, Asilo Sella, via Garibaldi 35, 07041 Alghero, SS, Italy*

ARTICLE INFO

Article history:
Received 1 May 2015
Received in revised form
12 December 2015
Accepted 18 December 2015
Handling Editor: R.E. Musafir
Available online 4 January 2016

Keywords:
Horn equation
Blockage identification
Inverse problems in acoustics
Numerical modelling
Reconstruction
Quasi-isospectral Sturm-Liouville operators

ABSTRACT

This paper proposes a new method for the reconstruction of the blockage area function in a symmetric duct by resonant frequencies under a given set of end conditions, *i.e.*, open–open or closed–closed ends. The analysis is based on the explicit determination of quasi-isospectral ducts, that is duct profiles which have the same spectrum as a given duct with the exception of a single eigenfrequency which is free to move in a prescribed interval. The analytical reconstruction was numerically implemented and tested for the detection of blockages. Numerical results show that the accuracy of identification increases with the number of eigenfrequencies used and that the reconstruction is rather stable with respect to the shape, the size and the position of the blockages.

1. Introduction

The identification of the cross-sectional area variation induced by the occurrence of blockages in a duct or in a pipe by using non-intrusive acoustic measurements is an important issue in several contexts. Applications range from detection of blockages in sodium cooled fast reactors [1] to diagnostic analysis of piped fluid systems. Other research works concern the use of classical acoustic methods based on impedance testing for the identification of cross-sectional area changes in vocal tracts or ear canals [2,3], although more efficient approaches have been recently developed for the analysis of these inherently dissipative systems, see, for example, [4,5]. Among advanced applications in the field, the research developed by Campbell and co-workers on the use of Acoustic Pulse Reflectometry for determining the internal dimensions of musical wind instruments [6] and for leak detection in tubular objects [7] should be also mentioned.

A commonly accepted approximation to the wave equation that governs the low-frequency sound propagation in a duct is Webster's horn equation [8]. The model considers the duct to be a slender hard-walled tube, lossless, and to have a rate of change of cross sectional area with the distance x along the tract that is sufficiently small, so that the sound pressure can be approximated by means of a longitudinal sound wave along the x-direction. For a sound pressure $p(x,t)$ varying harmonically in time with radian frequency ω, *i.e.*, $p(x,t)=u(x)e^{i\omega t}$, where i is the imaginary unit, the spatial propagation of the

* Corresponding author.
E-mail addresses: antonio.bilotta@unical.it (A. Bilotta), antonino.morassi@uniud.it (A. Morassi), emilio.turco@uniss.it (E. Turco).
[1] Tel.: +39 0432 558739; fax: +39 0432 558700.

http://dx.doi.org/10.1016/j.jsv.2015.12.038

longitudinal acoustic wave of small amplitude $u(x)$ is governed by the Webster horn operator

$$(A(x)u'(x))' + \lambda A(x)u(x) = 0, \quad \text{in } (0, L), \quad \lambda = \frac{\omega^2}{c_s^2}, \tag{1.1}$$

where $A = A(x)$ is the cross-sectional area, L is the duct length, c_s is the velocity of sound, and the primes denote x-differentiation. We refer to [9] for a rigorous derivation of Webster's horn equation based on the method of slow variation and ideas of matched asymptotic expansions. The boundary value problem is completed by assigning the conditions at the ends of the duct, namely the classical limit conditions $u' = 0$ for a closed end and $u = 0$ for an open end; see [9] for a rigorous asymptotic derivation of the boundary conditions and for a deep analysis of the boundary layer at the ends of slowly varying ducts. Under given (classical) end conditions at $x = 0$ and $x = L$, there exists a denumerable family of eigenvalues $\{\lambda_m\}_{m=1}^{\infty}$ (and eigenfrequencies, or resonant frequencies, ω_m) of Eq. (1.1), which form the *spectrum* of the duct. The corresponding non-trivial solutions u_m to Eq. (1.1) are the normal modes of vibration, or eigenfunctions, of the duct.

An important question is whether the shape $A(x)$ of a blockage perturbed duct can be determined from its resonant frequencies.

From the mathematical point of view, it has been shown that knowledge of two complete spectra, having specific asymptotic forms and satisfying specific interlacing conditions, correspondent to two different end conditions of the duct (*i.e.*, closed–closed and closed–open ends) uniquely determines the cross-sectional area $A(x)$, up to a multiplicative factor, see [10–12]. Knowledge or measurement of the doubly infinite set of eigenvalues is not realistic, since it is only for low frequencies that Webster's horn equation properly models the sound propagation in a duct. Therefore, in practical applications, the attention must be necessarily restricted to a set of lowest-order eigenvalues.

Reconstruction of the cross-sectional area from a finite number of (lower) eigenfrequency data has been carried out in the literature using perturbation analysis. On assuming that the unknown cross-sectional area $A(x)$ is a smooth perturbation of the uniform closed–open duct A_0 and that the logarithm of the normalized area function variation $\ln(A(x)/A_0)$ is band limited in frequency preserving only $2N$ cosine Fourier components, *e.g.*, $\ln\big(A(x)/A_0\big) = \sum_{j=1}^{2N} a_j \cos \frac{2jx}{L}$, Mermelstein [3] proved that the first-order change of the mth eigenfrequency of the closed–open duct and the change of the mth eigenfrequency of the closed–closed duct determines uniquely the $(2m-1)$th and the $2m$th Fourier coefficient of $\ln(A(x)/A_0)$, respectively.

The perturbation approach proposed by Mermelstein was extended by Wu and Fricke [13] to the detection of blockages in ducts by eigenfrequency measurements on closed–open and closed–closed end conditions. The identification procedure by Wu and Fricke was still based on the assumption that the blockages are a perturbation of the original duct, but, different from previous studies, the identification concerned with less regular cross-sectional coefficients, since the duct profile was assumed to be a piecewise-constant function. In spite of this weak regularity, the agreement between calculated and actual blockage area was good when half-wavelength of the eigenfrequency measured is greater than the length of the blockage.

The above reconstruction methods require the knowledge of a set of eigenvalues coming from spectra corresponding to two different end conditions. The requirement to modify a boundary condition (from closed–open to closed–closed, for example) in order to obtain information on a second spectrum, could be deemed and could represent a limit for the concrete application of the identification method. De Salis and Oldham [14] noticed that the completion of the finite Fourier expansion of $\ln(A(x)/A_0)$ is possible by using measurements under a single set of boundary conditions. They observed that eigenfrequencies for the closed–closed duct coincide with the antiresonant frequencies of the driven frequency response function measured at the open end of the closed–open duct. Estimate of antiresonant frequencies requires specific experimental and signal-processing strategies in order to locate the frequency with accuracy. In a subsequent paper, De Salis and Oldham [15] proposed high noise immunity maximum length sequence techniques to estimate accurately the locations of the antiresonant frequencies in the measured frequency response, and applied their method to identify blockages in a duct.

All the available results on the determination of the duct cross-sectional area from eigenfrequency measurements are founded on the assumption that the unknown cross-sectional area is a small perturbation of the intact (or initial) duct. However, the smallness of the perturbation is never stated in a quantitative way, *i.e.*, in terms of a suitable norm of the cross-sectional area change function, and this makes it difficult to determine the error in the reconstruction of the unknown coefficient. Based on the results by Wu and Fricke [13], for example, the perturbation analysis seems to be valid for blockage with change of area less than 50 percent of the intact area and blockage length less than $\frac{L}{4}$. In addition, since the reconstruction based on finite spectral data is not unique, it is not clear how large is the set of cross-sectional area coefficients $A(x)$ which share exactly all the first N eigenfrequencies coming from both the spectra under different end conditions. The above questions have motivated our research, and this paper is a contribution to this inverse problem in acoustics.

In this research we consider the problem of determining the geometry of a duct with blockages from a single spectrum. In the first part of the paper we show how to explicitly construct the cross-sectional area such that the duct has exactly the prescribed (measured) values of the first N eigenfrequencies belonging to a single spectrum obtained under either open–open or closed–closed end conditions. The analysis is developed for a symmetric duct, *e.g.*, a duct with cross-sectional area $A(x) \in C^2([0, L])$ such that $A(x) = A(L-x)$. In this case, the knowledge of a single full spectrum determines uniquely the shape profile, up to a multiplicative constant [16]. Our method is based on the determination of the so-called quasi-isospectral horn operators which have exactly the same spectrum as a given horn, with the exception of a single eigenfrequency which is free to move in a prescribed interval. The coefficient $A(x)$ and the normal modes can be constructed explicitly by means of