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# Axisymmetric diffraction of a cylindrical transverse wave by a viscoelastic spherical inclusion

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ABSTRACT

In this paper, the scattering and diffraction of a cylindrical transverse shear wave in a viscoelastic isotropic medium by a spherical heterogeneity is analytically solved. The waves are generated by the harmonic longitudinal oscillations of the cylinder walls. The spherical inclusion is located at the radial center of the cylinder and differs from the cylindrical material only in its complex shear modulus. Small amplitude motion is assumed, such that linear system theory is valid. By employing multi-pole expansions, the incident and scattered wave fields are each defined in both cylindrical and spherical coordinates allowing for the satisfaction of the boundary conditions at the surfaces of these multiply connected bodies. The solution involves an infinite sum of improper integrals, which are evaluated numerically. The wave field is determined for a hydrogel (alginate) bead suspended in a different hydrogel (agarose) that fills a glass test tube. Numerical examples showing the effect on displacement fields of varying the stiffness of the inclusion are presented. This solution is further validated with a finite element simulation showing excellent agreement with the analytic results.



## 1. Introduction

With the goal of non-destructively monitoring the growth of engineered tissue in beads of hydrogel scaffolding, Yasar et al. [1,2] pursued magnetic resonance (MR) elastography on the microscopic scale (μMRE), wherein high frequency displacements in the medium are discernible due to geometric focusing of the radially converging circular cylindrical, shear waves. Briefly, a glass cylindrical tube is filled with a viscoelastic solid and harmonically vibrated along its axis. Assuming a so-called welded contact between the solid and rigid tube wall, the mechanical oscillations of the latter induce radially convergent circular cylindrical shear waves in the former. The analytic solution to that problem is generally straightforward. Such is not the case if a different viscoelastic solid is embedded in the cylindrical medium that is a shape other than an infinite circular cylinder, e.g. a sphere, because the wave fields generated by each boundary cannot be expressed in the geometry of the other boundary. To satisfy the boundary conditions, each wave field must be mathematically described in the coordinate system of whatever boundary it touches. In the case of cylindrical and spherical waves, each is written in the other's coordinates with multipole expansions of appropriate Bessel functions. Over the past eight decades many researchers have studied mechanical problems involving a spherical heterogeneity centrally embedded in, or proximally

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external to, a cylindrical medium. Using Laplace's equation, Knight [3] studied the potential in a conducting cylinder with a spherical cavity. Kubenko et al. adopted this approach to solve the problems of pulsating [4] and potential [5] flow past a sphere in a rigid tube and thin elastic cylindrical shell, respectively, while Linton's work covers potential and Stokes flow around, and acoustic scattering by, a sphere in a cylinder [6]. Smythe solved for the velocity field of the flow around a sphere [7] and a spheroid [8] in a tube, starting with the vector potential functions to insert into the governing Laplace equation. Cai and Wallis [9] extended Smythe's work to include an infinite row of spheres in a tube.

Several authors have studied the phenomenon of waves that travel along the length of the cylinder. Ursell [10] considered a rigid tube filled with an acoustic medium and a centrally placed rigid sphere to demonstrate the existence of the so-called trapped modes, where normal velocity vanishes on the surfaces. Linton [11] extended this to include both soft and hard cylinders and spheres, while Zhuk et al. [12] considered the radiation force on the sphere. Lee [13] examined the scattering of torsional waves down an elastic cylinder by a central spherical cavity though Golovchan [14] had solved that problem for infinitely many stacked spherical cavities. Kubenko and Dzyuba wrote several papers where the spherical heterogeneity was the wave source, varying the type of cylindrical container and spherical inclusion. They include an oscillating and pulsating sphere in a rigid cylinder [15], a thin elastic cylindrical shell [16], and a thin elastic cylindrical shell immersed in an elastic medium [17,18]. Hasheminejad and Hosseini [19,20] solved for the case when the acoustic medium fills a cylindrical cavity in an infinite poroelastic medium. Hosseini and Namazi [21] applied the same coordinate transformations as above to solve for the case when the spherical wave source is external to an infinite circular poroelastic cylinder, i.e. a ball outside of a tube. Earlier, Li and Ueda [23] solved a very similar problem but assumed that the spherical waves could be modeled as a plane wave while Piquette [24] simplified matters by neglecting the radial component of the scattered wave. Adopting the integral equation formulation given by Ström [25], using the so-called T matrix method of Waterman [26], Olsson solved for a spherical cavity scattering a longitudinal wave propagating down a cylinder [27], waves from a perturbation at some point on the surface of the cylinder [28], as well as the scattering of elastic waves by a non-axisymmetric spherical cavity in a thick-walled pipe [29]. Finally, it is worth mentioning that this ball in a tube formalism extends far beyond acoustics, having been used by Kim et al. [30] to solve for the quantum scattering potential of atomic matter waves and by Otey and Fan [31] to solve for the electromagnetic heat transfer between a sphere and a plate. Presently, we will cast the problem in terms of differential, rather than integral, equations. Whereas Linton and Kubenko used the scalar Helmholtz equation, the viscoelastic nature of our problem calls for the vector Helmholtz equation. Golovchan [14] and Lee [13] both studied problems of elastic waves in cylinders with spherical obstacles, but they include only transverse, torsional waves, parallel to the boundaries. This produced no mode conversion, leaving only one type of wave to comprise the wave field. To our knowledge, then, our present study is the first to solve the problem of viscoelastic wave diffraction, in this particular geometry, under conditions that involve both longitudinal and transverse wave fields. In our formulation, the two media may differ mechanically in Poisson ratios, densities, and complex shear moduli, and geometrically in their radii. In our numeric example, however, we hold the Poisson ratios and densities to be equal that the materials differ only in their complex shear moduli. It is also important to note that since the sphere is modeled as being completely encased within the cylinder, the radius of the latter must exceed that of the former.

## 2. Problem formulation

Let there be given an infinitely long, rigid, circular cylindrical tube of inner radius $b$, filled with a viscoelastic solid. Embedded therein is a viscoelastic sphere (ball) of radius $a$ whose center is on the axis of the tube. The tube and the embedded ball are respectively described by cylindrical $(\rho, \varphi, z)$ and spherical $(r, \theta, \varphi)$ coordinates [33], shown in Fig. 1.

We choose these coordinate systems because they conveniently describe the spatial dependence of the field variables and the boundaries of the objects. The $z$-axis is aligned with the axis of the tube and contains the center of the spherical inclusion. The rigid tube harmonically oscillates along the cylindrical $z$-axis and we assume that the embedding material (medium 1) is in welded contact with both the spherical inclusion (medium 2) and the oscillating wall. The displacement of both media satisfies the vector Helmholtz equation,

$$\alpha_i^2 \nabla\nabla \cdot \mathbf{U} - \beta_i^2 \nabla \times \nabla \times \mathbf{U} + \omega^2 \mathbf{U} = 0 \tag{1}$$

where

$$\alpha_i^2 = (\lambda_i + 2\mu_i)\gamma_i^{-1} \quad (i = 1, 2), \tag{2a}$$

$$\beta_i^2 = \mu_i \gamma_i^{-1} \quad (i = 1, 2), \tag{2b}$$

and where $\nabla(\cdot)$, $\nabla \cdot (\cdot)$, and $\nabla \times (\cdot)$ are the well known gradient, divergence, and curl operations, respectively [32]. Here $\gamma_i$ is the density, and $\lambda_i$ and $\mu_i$ are the Lamé constants, the latter of which is the complex shear modulus, and $\omega$ is the angular frequency. The index $i$ is 1 or 2 to denote the medium, distinguishing one from the other. For convenience we will define $\lambda_i$ in terms of $\mu_i$ and the Poisson ratio, $\nu_i$, i.e. $\lambda_i = (2\mu_i \nu_i)/(1 - 2\nu_i)$. Since this analysis is ultimately to be used in the modeling of soft tissue, we have described the media as viscoelastic, which is why $\mu_i$ is a complex parameter. The real component, Re $\mu_i$, is the storage modulus and the imaginary component, Im $\mu_i$, is the loss modulus. The higher the magnitude of Im $\mu_i$, the higher is the damping. The limiting case of Im $\mu_i = 0$ describes a perfectly elastic material. The sign of Im $\mu_i$ matters. When it