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Motion of a torsion pendulum immersed in a linear viscous liquid. Influence of wave phenomena

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ABSTRACT

A cylindrical pendulum, which is suspended by an elastic rod or wire and immersed in a viscid liquid in a cylindrical container, can undergo rotating oscillations. The propagation velocity of vorticity perturbations in the gap between the cylinder and the container is assumed to be finite. This means that the acceleration of the liquid is characterized by not only the viscosity but also by a relaxation time constant. The propagation velocity of elastic torsional waves in the rod is assumed to be finite. The equations that describe the motion of such a complicated compound system are linear and have been solved in closed form. The solution shows that there is a considerable deviation between the exact solution and the simple quasi-steady solution. The most remarkable conclusion is that the classical quasi-steady solution for very weak damping is incompatible with the general solution. Propagation of elastic waves in the suspension rod and propagation of vorticity waves in the liquid have a great influence on the rotational motion of the pendulum. The purpose of this study is to formulate the criteria that make the classical quasi-steady analysis valid. The derived solution permits also measurement of viscosity and a conceivable relaxation coefficient of the liquid as well.

1. Introduction

The theory of non-steady rotational motion of a cylindrical torsion pendulum immersed in a cylindrical container, filled with an incompressible Newtonian liquid, and suspended by an elastic rod (or wire) has been outlined by Jones and Walters [4] and analyzed by Rehbinder and Apazidis [6]. These analyses are extensions of the simple quasi steady solution, presented in most textbooks on fluid mechanics. A remarkable property of the well-known quasi-steady solution is that it is incompatible with the more general one mentioned above if the damping is very weak. This fact seemed to have escaped the community of mechanics, a fact that is strange since the formation of Stokesian boundary layers is well known. If the damping is very weak, most of the liquid which is in the gap between the pendulum and the container is practically stagnant. Only a thin layer adjacent to the pendulum is moving. The extended nonsteady analysis is linear, implying that a solution is obtainable.

A torsion pendulum, which is damped by a viscous liquid, is often used for measurement of viscosity of the liquid. The evaluation of such measurements is usually based on quasi-steady theory. A device for this purpose is called a rotary viscometer and is described in elementary textbooks on fluid mechanics for engineering students, notably by Massey [5]. The evaluation of such a test can be founded on the solutions for free or forced oscillations. In the former case, the frequency and the logarithmic decrement are measured. The measurement is particularly simple if the damping is very weak. In the latter case, which is experimentally more attractive, the measurement is simple since it is only necessary to measure the

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List of symbols

c_r	speed of torsional waves in the rod
G_r	shear strength of the rod
k	torsional rigidity of the rod
J	moment of inertia of the pendulum
l_c	depth of the liquid in the container
L, l_r	lengths of the pendulum and the rod
M, M_r	masses of the pendulum and the rod
q_1, q_2	auxiliary variables
r	radial coordinate
s	variable in the complex plane
s_m	poles in the complex plane
t	time
T	torque acting at the pendulum
u	tangential velocity of the liquid
z	axial coordinate in the rod
α	damping coefficient
α_{crit}	critical damping coefficient for quasi-steady flow
β	damping coefficient
ε	relaxation time
ζ	dimensionless axial coordinate in the rod
θ	torsional displacement of the pendulum
ϑ	tangential displacement of the rod
κ	auxiliary function
μ	viscosity of the liquid
ξ	dimensionless radial coordinate in the liquid
ρ, ρ_p, ρ_r	densities of the liquid, the pendulum and the rod
σ	tangential stress in the liquid
τ	dimensionless time
Υ, Ψ	auxiliary functions
ω_0	inverse characteristic time
ω	imposed frequency
Ω	auxiliary function

phase shift and the damping of the oscillations. These methods presuppose that theory of the flow in the liquid is correct. If it is incorrect, viz. quasi-steady, the interpretation of experimental measurements for determination of viscosity becomes incorrect. This risk is particularly big if the damping is weak. For this reason the quasi-steady theory must be replaced by a non-steady theory.

Rehbinder and Apazidis [6] has pointed out two further linear phenomena that would be possible to include in the analysis. The two new phenomena to be considered are wave propagations.

The first wave phenomenon is propagation of radial vorticity waves in the liquid between the pendulum and the container, due to relaxation between the components of the stress tensor and the components of the rate-of-strain tensor. If the viscosity properties of a liquid are characterized not only by the ordinary viscosity coefficient but also by a relaxation time, Navier–Stoke's equation is not parabolic but hyperbolic. This means that vorticity waves with finite velocity will propagate in the liquid, although the damping is very strong. Such imagined waves are neglected for normal liquids. If an additive is solved in water, the properties of the mixture might be dramatically changed. Long chain polymers added to water may reduce or even remove turbulence. In an analogous way, it is conceivable that the propagation velocity of vorticity is finite, in the sense that it might be possible to measure it. In any case, measurement of a possible relaxation time cannot be evaluated unless a more general solution is available.

This kind of a relaxation time in a constitutive equation can be found in the theory of creep flow of an incompressible viscid liquid through a porous body with storativity. Normally it is assumed that the flux per unit area is proportional to pressure gradient. This assumption is called Darcy's law. Muscat [2] and Polubarinova-Kochina [1] have discussed the influence of a possible relaxation time in their celebrated textbooks, but concluded that the damping is so strong that no relaxation effect has to be considered. Pascal [3] has derived the solution of the 1D flow equation for a liquid that flows through a porous body including inertia relaxation. Introduction of a relaxation coefficient in Darcy's law is meaningful in the sense that it can be experimentally determined with fairly simple means, as Rehbinder [7] and [8] has shown. A 2D solution, corresponding to the draw down around a well in a confined aquifer has been derived by Löfqvist and Rehbinder [9]. The similarity between the propagation of water pressure in a liquid in a porous medium and the propagation of vorticity in a liquid described in this paper is striking. Another analogy is propagation of "heat waves" in a heat conducting body. It is discussed in chapter 6 in the comprehensive book by Morse and Feshback [10].

The second wave phenomenon that might have influence on the rotary motion of the pendulum is vertical propagation of linear shear waves in the suspension rod. The two suggested wave phenomena might interact. A possible relaxation time for vorticity waves in the liquid might be mixed up with the time constant of shear waves in the rod. This means that the present analysis is necessary, but by no means sufficient, for evaluation of viscosity tests with a rotary viscometer.

The purpose of this paper is to solve the coupled linear system of two partial differential equations and one ordinary differential equation together with appropriate boundary conditions. The three equations describe the coupled phenomena discussed above. The general solution can be used for evaluation of the viscosity properties of a liquid in a conventional rotary viscometer. The solution is necessary if the propagation velocity of vorticity is finite. Moreover, the purpose is to show the influence of shear waves in the suspension device. Finally the purpose is to show how the general solution degenerates into the well-known special cases. The background of the problem is presented in the paper by Rehbinder and Apazidis [6] and the reader is referred to this.

The present paper is organized in the following way. In the second section, the basic equations are presented. In the third section, the general problem, including appropriate boundary and initial conditions and the solution for free oscillations are