Journal of Sound and Vibration 334 (2015) 377–387

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

Mode shape sensitivity of two closely spaced eigenvalues

Rune Brincker [a,*], Manuel Lopez-Aenlle [b]

[a] Department of Engineering, Aarhus University, Edison Building, Finlandsgade 22, 8200 Aarhus N, Denmark
[b] Universidad de Oviedo, Campus de Gijón, s/n. Edificio 7, 33203, Gijón, Asturias, Spain

ARTICLE INFO

Article history:
Received 30 October 2013
Received in revised form
25 July 2014
Accepted 18 August 2014
Handling Editor: W. Lacarbonara
Available online 2 October 2014

ABSTRACT

In this paper the sensitivity of the mode shapes of two closely spaced eigenvalues are studied. It is well known that in case of repeated eigenvalues, the meaningful quantity is not the two individual mode shapes, but rather the subspace defined by the two mode shapes. Following the ideas of a principle that has been released for publishing recently denoted as the local correspondence (LC) principle, it is shown, that in the case of a set of two closely spaced eigenvalues, the mode shapes become highly sensitive to small changes of the system. However, if the two closely spaced eigenvalues have a reasonable frequency distance to all other eigenvalues of the system, then a linear transformation exists between the set of perturbed and unperturbed mode shapes describing the significant changes as a rotation in the initial subspace defined by the two mode shapes. Closed form solutions are given for general combined mass and stiffness perturbations, and it is shown that there is a smooth transition from the case of moderate sensitivity of the mode shapes towards the case of repeated eigenvalues where the sensitivity goes to infinite. In case of "nearly repeated eigenvalues" the perturbed set of mode shapes can be found by solving a special eigenvalue problem for the two closely spaced eigenvalues. The theory is illustrated and compared with the exact solution for a simple 3 dof system.

1. Introduction

Closely spaced modes appear in many practical applications. For instance a simple free–free beam with double symmetric cross section will in theory have lots of repeated poles because the bending modes around the two axis of bending will be exactly equal. In practice because small deviations in geometry and mass/stiffness distributions will always be present, all bending modes will appear in sets of two closely spaced modes. If the cross sections are different, then the two sets of bending modes will separate, but often it will appear that some bending mode will be closely spaced to some other bending or torsion mode. Similar conclusions can be drawn for plates and for nearly all structures in practice.

The increased sensitivity of mode shapes to small changes of the system in case of closely spaced eigenvalues is not well understood and not specifically much mentioned in the literature of structural dynamics even though the observation appear indirectly in the original treatment of the problem in the theoretical papers by Fox and Kapoor [1] and Nelson [2].

In this introduction we shall revisit the well-known properties of repeated eigenvalues, then consider a few of the remarks in the literature of structural dynamics on the problem of closely spaced eigenvalues, and then take a short look at

* Corresponding author.
E-mail addresses: rub@science.au.dk (R. Brincker), aenlle@uniovi.es (M. Lopez-Aenlle).

http://dx.doi.org/10.1016/j.jsv.2014.08.015
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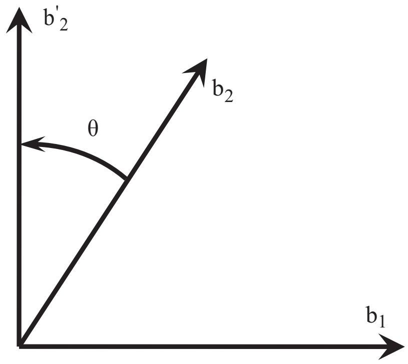

Fig. 1. In case of repeated eigenvalues, the corresponding mode shapes are not necessarily orthogonal, but one of the vectors might be chosen for the basis – in this case $\mathbf{b}_1$ – and the other one – in this case $\mathbf{b}_2$ – is then rotated in the subspace defined by $\mathbf{b}_1$, $\mathbf{b}_2$ until it is perpendicular to $\mathbf{b}_1$.

what has been done in numerical mathematics and in quantum mechanics where the sensitivity of similar eigenvalue problems have been studied.

As we shall see from the main results of this paper, when a system with two closely spaced eigenvalues is perturbed the associated mode shapes are mainly rotating in their initial subspace. We are here assuming that there is a reasonable frequency distance to all other eigenvalues of the system so that the influence of other eigenvalues can be ignored. These assumptions are not always fulfilled in practical applications, for instance we might have situations with three closely spaced modes or cases where the distance to the other modes cannot be completely neglected. In such cases the problem needs to be studied using the more general theory as for instance presented in Brincker et al. [10]. However, if we accept the assumptions, the two considered modes should be understood more as defining a subspace than as individual vectors.

The case of closely spaced eigenvalues has inherited this important property from the case of repeated eigenvalues, and therefore it is useful to revisit this well-known case as an introduction to the subject of this paper. Considering the theory of repeated eigenvalues we will limit the analysis to the simple classical eigenvalue problem related to the un-damped case of a general dynamic system with N degrees of freedom (dof's)

$$\mathbf{M}^{-1}\mathbf{K}\mathbf{b} = \omega^2\mathbf{b} \tag{1.1}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{K}$ is the stiffness matrix, ω^2 is one of the positive and real eigenvalues, where ω is denoted the natural frequency, and $\mathbf{b}$ is the corresponding real valued eigenvector, also denoted the mode shape. The natural frequencies and the mode shapes are found by the eigenvalue decomposition $\mathbf{M}^{-1}\mathbf{K} = \mathbf{B}[\omega_n^2]\mathbf{B}^{-1}$ where $\mathbf{B} = [\mathbf{b}_n]$ is a matrix of eigenvectors and $[\omega_n^2]$ is a diagonal matrix holding the eigenvalues.

The well-known orthogonality properties of the mode shapes are easily verified writing Eq. (1.1) for two different modes with the natural frequencies ω_n, ω_m and the mode shapes $\mathbf{b}_n, \mathbf{b}_m$. Multiplying each of the equations with the other transposed mode shape from the left, subtracting the two equations and using that $\mathbf{b}_n^T\mathbf{K}\mathbf{b}_m = \mathbf{b}_m^T\mathbf{K}\mathbf{b}_n$ we get

$$(\omega_n^2 - \omega_m^2)\mathbf{b}_n^T\mathbf{M}\mathbf{b}_m = 0 \tag{1.2}$$

Thus we conclude that if the two eigenvalues are different, the inner product $\mathbf{b}_n^T\mathbf{M}\mathbf{b}_m = 0$. This leads directly to the well-known orthogonality equations $\mathbf{B}^T\mathbf{M}\mathbf{B} = [m_n]$ and $\mathbf{B}^T\mathbf{K}\mathbf{B} = [k_n]$, where the diagonal matrices $[m_n]$ and $[k_n]$ holds the modal masses and the modal stiffness's respectively.

In case some eigenvalues are equal the associated eigenvectors can either still exist or degenerate into a single eigenvector, see Bernal [3]. In the latter case the matrix $\mathbf{M}^{-1}\mathbf{K}$ is defective (will not have a complete basis of eigenvectors) and we shall not consider that case here. Let us say that we consider a case where the two eigenfrequencies ω_1, ω_2 corresponding to the mode shapes $\mathbf{b}_1$, $\mathbf{b}_2$ are identical, thus $\omega_1 = \omega_2 = \omega$. From Eq. (1.2) we see that the orthogonality between the two mode shapes $\mathbf{b}_1$,$\mathbf{b}_2$ is no longer assured, as the condition given by Eq. (1.2) is always satisfied when $\omega_1 - \omega_2 = 0$. However, any linear combination of the two mode shapes $\mathbf{b} = t_1\mathbf{b}_1 + t_2\mathbf{b}_2$ is also an eigenvector because from the eigenvalue problem (1.1) we have that $\mathbf{M}^{-1}\mathbf{K}(t_1\mathbf{b}_1 + t_2\mathbf{b}_2) = \omega^2\mathbf{b}$.

We can define a new set of eigenvectors such that the set satisfies the orthogonality equations. For instance we can choose the new set of eigenvectors as $\mathbf{b}_1, \mathbf{b}'_2$ where the vector $\mathbf{b}'_2 = [\mathbf{b}_1, \mathbf{b}_2]\mathbf{t} = \mathbf{B}\mathbf{t}$ and orthogonal to $\mathbf{b}_1$, i.e. such that

$$\mathbf{b}_1^T\mathbf{M}\mathbf{B}\mathbf{t} = 0 \tag{1.3}$$

and we can define the transformation by the angle θ setting $\mathbf{t}^T = \{\cos\ \theta,\ \sin\ \theta\}$. In this light we can say that we rotate the vector $\mathbf{b}_2$ in the subspace defined by $\mathbf{b}_1$,$\mathbf{b}_2$ until $\mathbf{b}_2$ is perpendicular to $\mathbf{b}_1$, see Fig. 1.

Calculating the row vector $\{m_1, m_2\} = \mathbf{b}_1^T\mathbf{M}\mathbf{B}$ and using Eq. (1.3) defines a possible solution as $\theta = \arctan(-m_1/m_2)$.[1]

[1] It should be noted that this procedure does not secure that the length (scaling) of the rotated vector is kept unchanged.