Journal of Sound and Vibration 333 (2014) 2554–2567

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

A comparative study of the harmonic balance method and the orthogonal collocation method on stiff nonlinear systems

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ARTICLE INFO

Article history:
Received 7 May 2013
Received in revised form
21 January 2014
Accepted 24 January 2014
Handling Editor: L.N. Virgin
Available online 21 February 2014

ABSTRACT

The high-order purely frequency-based harmonic balance method (HBM) presented by Cochelin and Vergez (2009) [1] and extended by Karkar et al. (2013) [2] now allows to follow the periodic solutions of regularized non-smooth systems (stiff systems). This paper compares its convergence property to a reference method in applied mathematics: orthogonal collocation with piecewise polynomials. A first test is conducted on a nonlinear smooth 2 degree-of-freedom spring mass system, showing better convergence of the HBM. The second test is conducted on a one degree-of-freedom vibro-impact system with a very stiff regularization of the impact law. The HBM continuation of the nonlinear mode was found to be very robust, even with a very large number of harmonics. Surprisingly, the HBM was found to have a better convergence than the collocation method for this vibro-impact system.

1. Introduction

The literature is crudely lacking comparative studies between purely time-based and purely frequency-based numerical methods for computing periodic solutions of nonlinear dynamical systems. This is especially true regarding the behavior of numerical methods when addressing stiff mechanical systems like impacting oscillators (vibro-impact system). This paper aims at comparing two such methods, in a general framework where one wishes to compute families, or branches, of periodic solutions of such systems using a numerical continuation algorithm. This is an important issue in many scientific fields and engineering applications.

In the literature, various numerical methods have been proposed to directly compute such periodic solutions [3–6] without resorting to numerical time integration techniques, which provide stable periodic solutions only as a limit set and can be very time-consuming, especially for stiff systems. These direct numerical methods are generally classified into two main categories referred to as the frequency domain approach and the time domain approach.

The emblematic method for the frequency domain approach is the so-called harmonic balance method (HBM) which relies on the representation of the periodic orbit by a truncated Fourier series for the unknown state variables. The HBM substitutes the series into the nonlinear governing equation, collecting terms with the same harmonic number and dropping terms with harmonic numbers not in the Fourier series. This leads to solving an algebraic system for the Fourier coefficients which balances harmonics. HBM is better presented as a weighted residual method: it is a Galerkin method with Fourier basis and Fourier test functions, and for which convergence has been established for instance by Urabe [7]. Note that some authors describe the HBM as unpractical or cumbersome, as it implies analytical derivation of the relations between Fourier

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http://dx.doi.org/10.1016/j.jsv.2014.01.019

coefficients involved in the nonlinear terms. However, previous works have shown that first, most nonlinearities can be recast as quadratic polynomials using additional variables; and second, in the quadratic case it is very easy to automate this analytical work (see [2]). Many variations of the basic HBM exist, such as the alternating frequency time-HBM [8], the multi-HBM [9], the incremental HBM [10], and the adaptive HBM [11]. Some of these variations improve usability, performance, or robustness. Some adapt to situations such as non-smooth systems [12] or delay systems for example.

Two emblematic methods for the time domain approach are the shooting method [5,4] (not considered here, as it does use time integration) and the global finding of periodic orbits using a boundary value approach [6]. The orthogonal collocation with piecewise polynomials (later referred to as collocation) belongs to the second one: the periodic orbit is divided into mesh intervals, the unknown state variables are represented by polynomials on each interval and the governing equations are collocated at Gauss points. This collocation method may also be seen as a weighted residual method (in this case, a Petrov–Galerkin method) with piecewise polynomial basis and Dirac test functions, and many variations exist. To end, it is worth to note that a third category could have been introduced for trigonometric collocation methods [13], or similarly the high-dimensional HBM [14] which, despite the name, is more a collocation method than an HBM as shown by [15]. Methods belonging to the latest category are once more weighted residual methods, but with Fourier basis and Dirac test functions.

Today, HBM is very popular in electrical engineering (electronic circuit) and in mechanical engineering (structural dynamics, rotor dynamics) while the collocation method is very popular for biological systems, population systems, chemical reactions analysis and more generally for applied mathematics (the collocation method is for instance implemented in the AUTO software [16], as well as in the MATCONT package [17], a MATLAB [18] toolbox). So, it seems that the choice between the frequency domain approach and the time domain approach is not only a question of performance and ease of implementation, but also a question of experience inside a scientific field. As stated earlier, the literature lacks comparisons between these two categories of methods: typical papers describe a numerical method and demonstrate its performance on selected representative examples, but comparisons are seldom performed.

The present study compares the high-order purely frequency HBM presented in [1] and extended in [2], with the piecewise polynomial collocation method. For this, a still challenging mechanical problem is chosen: periodic solutions continuation of a regularized vibro-impact system, that is, nonlinear mode calculations of a non-smooth system. The comparison is carried out using the asymptotic numerical method (ANM) for the continuation and each of the aforementioned methods for the discretization. Because many variations exist for each category of methods, a few conditions have to be fixed for the comparison. Hereafter, focus is brought to the accuracy of the solution versus the number of unknowns in the algebraic system. Second, the comparison is limited to small size dynamical systems. Third, no adaptive mesh is used for the collocation and no harmonic selection is used for the HBM. Within this framework, and despite the common wisdom that would advise against using the HBM for systems with stiff nonlinearities, the HBM achieves a better convergence rate than the collocation, even for very stiff problems.

The paper is organized as follows: in Section 2, the harmonic balance method and the orthogonal collocation with piecewise polynomials method are reviewed, and their theoretical convergence rates are recalled. In Section 3, their convergences are compared on a toy-model composed of a slightly nonlinear, one-mass, two-spring plane system (representative of shells under large strain), as well as their efficiency for calculating periodic orbit families, when coupled with the ANM continuation technique. Then, in Section 4, the same methodology is used to compare both approaches on a highly nonlinear system: an impacting oscillator with exponential restoration force. The conclusions of this comparative study are outlined in the last section.

2. Discretization methods for periodic orbits

In this section, the two methods that are used here for solving the periodic boundary-value problem that consists in finding a periodic solution of a given autonomous, nonlinear dynamical system are briefly reviewed. The problem is to find $\mathbf{Y}:\mathbb{R}\rightarrow\mathbb{R}^n$ and its associated period $T\in\mathbb{R}_+$ such that $\forall t\in\mathbb{R}$

$$\mathbf{Y}'(t)=f(\mathbf{Y}(t)) \tag{1}$$

$$\mathbf{Y}(t)=\mathbf{Y}(t+T) \tag{2}$$

where f is a nonlinear application $\mathbb{R}^n\rightarrow\mathbb{R}^n$ and the prime sign denotes the time derivative.

The general principle of a spectral method is to choose a vector-space $\mathbb{E}$ in which one wishes to approximate the solutions, together with a basis of this space: the representation functions $\{\phi_i(t)\}$. Then one writes a number of algebraic equations resulting from the orthogonalization of the residue $\mathcal{R}(\mathbf{Y}(t))=\mathbf{Y}'(t)-f(\mathbf{Y}(t))$ to this vector-space, with respect to the corresponding scalar product. This second step is usually carried out by canceling out the projection of the system's ordinary differential equations onto a set of functions, usually but not necessarily a basis of $\mathbb{E}$: the test (or weighting) functions. The reader is referred to Orszag [19] and Karniadakis and Sherwin [20] for the original works and a recent reformulation on spectral and pseudo-spectral methods.