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Higher-order stochastic averaging to study stability of a fractional viscoelastic column

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ABSTRACT

Stochastic stability of a fractional viscoelastic column axially loaded by a wideband random force is investigated by using the method of higher-order stochastic averaging. By modelling the wideband random excitation as Gaussian white noise and real noise and assuming the viscoelastic material to follow the fractional Kelvin–Voigt constitutive relation, the motion of the column is governed by a fractional stochastic differential equation, which is justifiably and uniformly approximated by an averaged system of Itô stochastic differential equations. Analytical expressions are obtained for the moment Lyapunov exponent and the Lyapunov exponent of the fractional system with small damping and weak random fluctuation. The effects of various parameters on the stochastic stability of the system are discussed.

1. Introduction

There is a continual interest in the study of dynamic stability of a viscoelastic column [1,2], partly due to the fact that many structural elements can be studied as columns (shafts, pillars, etc.) and the study of vibrating columns yields interesting insights on the behavior of a much wider class of systems. Dynamic stability of columns under deterministic loads has been intensively studied [3,4]. At the same time, stochastic stability attracts equal attention. A large number of results were obtained when the loading is modelled as a Gaussian white noise because of the availability of mathematical theory, such as the Itô calculus, in dealing with white noise processes. Sufficient conditions for the almost sure stability of the solutions of differential equations under random perturbations of their parameters were obtained by Khasminskii [5]. Dynamic stability of a Kelvin-viscoelastic column subjected to a time-varying axial load was considered and sufficient stability criteria for columns were also established [6]. Potapov [7] proposed an approach for the analysis of the stability of different stochastic systems using both analytical and numerical methods. Drozdov and Kolmanovskii systematically studied the stability of viscoelastic systems [8], such as stability in mean square sense, stability by using the direct Lyapunov method, and sufficient conditions for the almost sure stability for viscoelastic rods.

However, the estimations of stability boundaries from these sufficient conditions are usually rather rough [7]. More accurate results can be obtained for systems with small damping and weak stochastic fluctuation, by using an asymptotic method known as the method of stochastic averaging, which was proposed by Stratonovich [9] and proved rigorously by Khasminskii [10] in a limit theorem for stochastic differential equations. Response and Lyapunov exponents of a SDOF strongly nonlinear stochastic system with light damping modeled by a fractional derivative were studied by using stochastic

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averaging [11]. Ariaratnam [12] applied the method of stochastic averaging to study stochastic stability of linear viscoelastic systems. Huang and Xie extended Ariaratnam's approach to second-order stochastic averaging [13], finding that the second-order averaging results agree better with those estimated by Monte Carlo simulation than the first-order averaging.

This paper attempts to investigate whether there is a further improvement by using higher-order stochastic averaging method to study stability of a viscoelastic system, which is obviously an extension of [14] from lower-order averaging to higher-order averaging. The paper is also an extension of [13] from ordinary viscoelasticity to fractional viscoelasticity. Furthermore, the method of stochastic averaging is justified for applications in fractional stochastic differential equations.

To the authors' best knowledge, no results on higher-order stochastic averaging can be found in the literature, although higher-order deterministic averaging was once reported [15]. The reason is apparently due to the facts that higher-order stochastic averaging requires far more complex mathematical manipulations than higher-order deterministic averaging, and it is difficult to implement even by using symbolic computation software.

2. Formulation

Consider stability problem of a column of uniform cross section under dynamic axial compressive load as in [14]. The equation of motion for this column is given by [16]

$$\frac{\partial^2 M(x,t)}{\partial x^2} = \rho A \frac{\partial^2 v(x,t)}{\partial t^2} + \beta_0 \frac{\partial v(x,t)}{\partial t} + F(t) \frac{\partial^2 v(x,t)}{\partial x^2}, \tag{1}$$

where ρ is the mass density per unit volume of the column, A is the cross-sectional area, $v(x,t)$ is the transverse displacement of the central axis, β_0 is the damping constant. The moment $M(x,t)$ at the cross-section x and the geometry relation are

$$M(x,t) = \int_A \sigma(x,t) z \, \mathrm{d}A, \quad \varepsilon(x,t) = -\frac{\partial^2 v(x,t)}{\partial x^2} z. \tag{2}$$

The viscoelastic material is supposed to follow the fractional Kelvin–Voigt constitutive model [14]

$$\sigma(t) = \left[E + \eta \cdot {}_0^{RL}\mathcal{D}_t^{\mu}\right] \varepsilon(t), \quad {}_0^{RL}\mathcal{D}_t^{\mu}[\varepsilon(t)] = \frac{1}{\Gamma(1-\mu)} \left[\frac{\varepsilon(0)}{t^{\mu}} + \int_0^t \frac{\varepsilon'(\tau)}{(t-\tau)^{\mu}} \, \mathrm{d}\tau\right], \tag{3}$$

where $0 < \mu \leq 1$ and $\Gamma(\mu)$ is the gamma function

$$\Gamma(\mu) = \int_0^{\infty} \mathrm{e}^{-t} t^{\mu-1} \, \mathrm{d}t,$$

and ${}_0^{RL}\mathcal{D}_t^{\mu}[\varepsilon(t)]$ is the Riemann–Liouville fractional derivative of function $\varepsilon(t)$ [17].

Substituting Eq. (3) into (2) and then to (1) yields

$$\rho A \frac{\partial^2 v(x,t)}{\partial t^2} + \beta_0 \frac{\partial v(x,t)}{\partial t} + EI \frac{\partial^4 v(x,t)}{\partial x^4} + I\eta \cdot {}_0^{RL}\mathcal{D}_t^{\mu} \left[\frac{\partial^4 v(x,t)}{\partial x^4}\right] + F(t) \frac{\partial^2 v(x,t)}{\partial x^2} = 0. \tag{4}$$

If the column is simply supported, the transverse deflection can be expressed as

$$v(x,t) = \sum_{n=1}^{\infty} q_n(t) \sin \frac{n\pi x}{L}. \tag{5}$$

Substituting Eq. (5) into (4) leads to the equations of motion

$$\ddot{q}_n(t) + 2\beta \dot{q}_n(t) + \omega_n^2 \left[1 - \frac{F(t)}{P_n} + \frac{\eta}{E} {}_0^{RL}\mathcal{D}_t^{\mu}\right] q_n(t) = 0, \tag{6}$$

where

$$\beta = \frac{\beta_0}{2\rho A}, \quad \omega_n^2 = \frac{EI}{\rho A} \left(\frac{n\pi}{L}\right)^4, \quad P_n = EI \left(\frac{n\pi}{L}\right)^2. \tag{7}$$

If only the nth mode is considered, and the damping, the viscoelastic effect, and the amplitude of load are all small, the equation of motion can be written as, by introducing a small parameter $0 < \varepsilon \ll 1$,

$$\ddot{q}(t) + 2\varepsilon\beta \dot{q}(t) + \omega^2 \left[1 + \varepsilon \xi(t) + \varepsilon \tau_{\varepsilon} \cdot {}_0^{RL}\mathcal{D}_t^{\mu}\right] q(t) = 0, \quad \tau_{\varepsilon} = \frac{\eta}{E}. \tag{8}$$

Mathematically, random excitations can be modelled as stochastic processes. The Gaussian white noise process is a weakly stationary process with constant cosine power spectral density $S(\omega) = \sigma^2$ and sine power spectral density $\Psi(\omega) = 0$ over the entire frequency range. Another important wideband noise is called real noise, which is often characterized by an Ornstein–Uhlenbeck process and is given by $\mathrm{d}\xi(t) = -\alpha \xi(t) \, \mathrm{d}t + \sigma \, \mathrm{d}W(t)$ with cosine and sine power spectral density

$$S(\omega) = \frac{\sigma^2}{\alpha^2 + \omega^2}, \quad \Psi(\omega) = \frac{\omega \sigma^2}{\alpha(\alpha^2 + \omega^2)}, \tag{9}$$

where $W(t)$ is a standard Wiener process [16].