



Introducing a Green–Volterra series formalism to solve weakly nonlinear boundary problems: Application to Kirchhoff's string

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ABSTRACT

This paper introduces a formalism which extends that of “Green's function” and that of “the Volterra series”. These formalisms are typically used to solve, respectively, linear inhomogeneous space–time differential equations in physics and weakly nonlinear time–differential input-to-output systems in automatic control. While Green's function is a space–time integral kernel which fully characterizes a linear problem, the Volterra series expansions involve a sequence of multi-variate time integral kernels (of convolution type for time-invariant systems). The extension proposed here consists in combining the two approaches, by introducing a series expansion based on multi-variate space–time integral kernels. This series allows the representation of the space–time solution of weakly nonlinear boundary problems excited by an “input” which depends on space and time.

This formalism is introduced on and applied to a nonlinear model of a damped string that is excited by a transverse mass force $f(x, t)$. The Green–Volterra kernels that solve the transverse displacement dynamics are computed. The first-order kernel exactly corresponds to Green's function of the linearized problem. The higher order kernels satisfy a sequence of linear boundary problems that lead to (both) analytic closed-form solutions and modal decompositions. These results lead to an efficient simulation structure, which proves to be as simple as the one based on the Volterra series, that has been obtained in a previous work for excitation forces with separated variables $f(x, t) = \phi(x)f_{\text{tot}}(t)$. Numerical results are presented.

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1. Introduction

Sound synthesis based on physical models makes use of dynamical models of resonators that must be accurate enough to be realistic. Many models are available for strings, plates, pipes (see e.g. [1–10]). In addition to basic (conservative) wave propagation, these models include some relevant second-order phenomena such as nonlinearities that are responsible for the timbre variation with respect to the nuance (from pp to ff) and damping.

In this context, simulation methods are also an important issue. The Volterra series formalism [11] provides a convolution-type solution for “weakly nonlinear systems”, from which simple simulation structures can be derived and are available for bounded input signals. More precisely, it extends linear filtering (impulse response and transfer

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function) to the case of systems with distortions, the dynamics of which can be represented by a sum of multiple convolutions (multi-variate kernels and transfer kernels). For this reason, this tool has been widely used in e.g. signal processing, automatic control, electronics, mechanics (see e.g. [12–20]).

In a previous work [21], a nonlinear equation of a damped string has been solved using the Volterra series in the case where the string is excited by a force $f_{\text{tot}}(t)$ which is spatially distributed by a time-invariant function $\phi(x)$. Thus, a time-domain simulation of the transverse displacement $u(x, t)$ has been derived from the truncated series, which provides accurate results for all signal input $f_{\text{tot}}(t)$ with sufficiently small amplitudes. In practice, for sound synthesis issues (bowed string, pinched string, etc), the main limitation of this solution is due to the time-invariance of the spatial distribution ϕ . In the linear case, for a general space–time transverse mass force $f(x, t)$, this problem is solved by using Green's function which fully characterizes the considered system.

In this paper, we introduce a “Green–Volterra” formalism which extends both that of “Green's function” and that of “the Volterra series”. Applying this formalism to the string allows one to compute its dynamics whatever the excitation force $f(x, t)$ may be. The solution is given by a series expansion composed of space–time integrals involving the (computable) “Green–Volterra” kernels. The first term of the series expansion exactly corresponds to Green's function solution. Higher order terms introduce nonlinear dynamics contributions.

The paper is organized as follows. Section 2 introduces the nonlinear string model. It recalls results based on Green's function for the linearized problem (Section 2.2) and based on the Volterra series for a particular class of excitations (Section 2.3). Section 3 introduces the formalism of Green–Volterra series. Then, this formalism is used to solve the original problem in Section 4. A simulation is deduced in Section 5. Finally, Section 6 develops conclusions and perspectives.

2. Problem statement

2.1. Model under consideration

Consider the dimensionless nonlinear Kirchhoff model of the transverse vibrations of a damped string [2] excited by a force f distributed on $\Omega =]0, 1[$ given by, for all $(x, t) \in \Omega \times \mathbb{R}_+$,

$$\partial_t^2 u(x, t) + 2\alpha \partial_t u(x, t) - \left(1 + \varepsilon \left[\int_{\Omega} (\partial_x u(x, t))^2 dx \right]\right) \partial_x^2 u(x, t) = f(x, t), \quad (1)$$

with Dirichlet boundary conditions and zero initial conditions

$$\forall t > 0, \quad u(x=0, t) = 0 \quad \text{and} \quad u(x=1, t) = 0, \quad (2)$$

$$\forall x \in \Omega, \quad u(x, t=0) = 0 \quad \text{and} \quad \partial_t u(x, t=0) = 0. \quad (3)$$

Coefficients ε and α are positive. They quantify the effects of the nonlinearity (due to the variation of tension) and of the fluid damping (due to viscosity, see e.g. [22, p. 194]).

2.2. Linearized problem: Green's function solution

The linearized version of (1)–(3) is obtained for $\varepsilon = 0$. It has the form

$$L_{x,t}[u] = f, \quad (4)$$

where $L_{x,t}$ is the linear differential operator $\partial_t^2 + 2\alpha \partial_t - \partial_x^2$ over the domain $(x, t) \in \Omega \times \mathbb{R}_+$ associated with Dirichlet boundary conditions and zero initial conditions. This standard well-posed problem has been extensively studied.

In general, boundary linear problems governed by Eq. (4) can be solved using Green's function formalism [23,24] (see also e.g. [22,25] for applications in musical acoustics): the solution is the superposition of all local contributions of f , given by the integral

$$u(x, t) = \int_{\Omega \times \mathbb{R}_+} g(x, t; \xi, \tau) f(\xi, \tau) d\xi d\tau, \quad (5)$$

where Green's kernel g denotes the solution of Eq. (4) excited by the Dirac pulse centered at (ξ, τ) , that is,

$$f(x, t) = [\delta_{\xi} \otimes \delta_{\tau}](x, t) = \delta(x - \xi) \delta(t - \tau), \quad (6)$$

where $\otimes$ denotes the tensor product of two distributions [26, Chapter 3.1].

Note that this problem is time invariant but not space invariant so that $g(x, t; \xi, \tau) = g(x, 0; \xi, \tau - t)$. Indeed, denote by $T_{(\tau, \xi)}$ the translation operator of duration τ and length ξ . If (f, u) is a solution, it appears that $(T_{(\tau, 0)} f, T_{(\tau, 0)} u)$ is also a solution (time-invariance), because $T_{(\tau, 0)}$ commutes with all the operators of the partial differential equation, whereas $(T_{(0, \xi)} f, T_{(0, \xi)} u)$ is not. Hence, Eq. (5) becomes a convolution w.r.t. time, that is,

$$u(x, t) = \int_{\Omega \times \mathbb{R}_+} \underline{g}(x; \xi, \tau) f(\xi, t - \tau) d\xi d\tau \quad \text{where} \quad \underline{g}(x; \xi, \tau) = g(x, 0; \xi, -\tau).$$

denotes the convolution-type Green's kernel.

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