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Short Communication

Iteration method solutions for conservative and limit-cycle $x^{1/3}$ force oscillators

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Abstract

An iterative technique is used to calculate a higher-order approximation to the periodic solutions of a conservative oscillator for which the elastic force term is proportional to $x^{1/3}$. The related van der Pol-type limit-cycle oscillator is also studied.

The purpose of this Short Communication is to calculate a higher-order approximation to the periodic solutions of the following differential equations [1–3]:

$$\ddot{x} + x^{1/3} = 0, \tag{1}$$

$$\ddot{x} + x^{1/3} = -\varepsilon(1 - x^2)\dot{x}, \tag{2}$$

using an iteration technique derived by Mickens [4]. These equations represent a new class of nonlinear oscillating systems [1]. The work presented here extends previous results given in Mickens [1–3] which relied primarily on the method of harmonic balance [5] as the tool for determining the oscillatory solutions.

The details of the iteration technique are given in Mickens [4]; consequently, only an outline of the method is required here.

The nonlinear oscillator equation is assumed to take the form

$$\ddot{x} + g(x) = \varepsilon f(x, \dot{x}), \quad x(0) = A, \;\; \dot{x}(0) = 0, \tag{3}$$

where ε is a positive parameter and the functions $g(x)$ and $f(x, \dot{x})$ are assumed to satisfy the conditions

$$g(-x) = -g(x), \quad f(-x, -\dot{x}) = -f(x, \dot{x}). \tag{4}$$

Eq. (3) can be rewritten as

$$\ddot{x} + \Omega^2 x = G(x, \dot{x}), \tag{5}$$

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where the constant Ω^2 is to be determined later and $G(x, \dot{x})$ is given by the expression

$$G(x, \dot{x}) \equiv \Omega^2 x - g(x) + \varepsilon f(x, \dot{x}). \tag{6}$$

The iteration scheme defines the $(k+1)$-approximation to the solution of Eq. (5) as

$$\ddot{x}_{k+1} + \Omega^2 x_{k+1} = G(x_k, \dot{x}_k) + G_x(x_k, \dot{x}_k)(x_k - x_{k-1}) + G_{\dot{x}}(x_k, \dot{x}_k)(\dot{x}_k - x_{k-1}), \tag{7}$$

where $k = 0, 1, 2, \ldots,$

$$G_x \equiv \frac{\partial G}{\partial x}, \quad G_{\dot{x}} \equiv \frac{\partial G}{\partial \dot{x}}, \tag{8}$$

and the initiation or starting solutions are

$$x_{-1}(t) = x_0(t) = A\cos(\Omega t), \tag{9}$$

$$x_{k+1}(0) = A, \quad \dot{x}_{k+1}(0) = 0. \tag{10}$$

The angular frequency, Ω, is calculated anew at each stage of the iteration procedure by demanding that the right-hand side of Eq. (7) contains no terms giving rise to secular terms in the complete solution of Eq. (7) with initial conditions stated in Eq. (10).

For the oscillator modeled by Eq. (1), it follows that

$$G(x, \dot{x}) = \Omega^2 x - x^{1/3}. \tag{11}$$

Note that $G_{\dot{x}}(x, \dot{x}) = 0$, and

$$G_x(x, \dot{x}) = \Omega^2 - \left(\frac{1}{3}\right)\frac{1}{x^{2/3}}. \tag{12}$$

Thus, G_x has a singularity at $x = 0$. However, what appears, for $k = 0$, in Eq. (7) is the expression $G_x(x_0, \dot{x}_0)(x_0 - x_{-1})$ which when properly evaluated, using Eq. (9), gives the result

$$G_x(x_0, \dot{x}_0)(x_0 - x_{-1}) = 0. \tag{13}$$

This means that the differential equation to be solved for $x_1(t)$ is

$$\ddot{x}_1 + \Omega^2 x_1 = \Omega^2 A\cos(\Omega t) - [A\cos(\Omega t)]^{1/3}. \tag{14}$$

At this point, there are two things to note. First, the iteration scheme cannot be extended for Eq. (1) to calculate $x_k(t)$ for $k \geqslant 2$. This is because the singularities occurring on the right-hand side of Eq. (7) cannot be eliminated. Second, a Fourier series representation is needed for $(\cos\theta)^{1/3}$ for the calculation of $x_1(t)$ to proceed.

The Fourier series for $(\cos\theta)^{1/3}$ has been calculated [6] and is given by

$$(\cos\theta)^{1/3} = \sum_{n=0}^{\infty} a_{2n+1}\cos(2n+1)\theta, \tag{15}$$

$$a_{2n+1} = \frac{3\Gamma(\frac{7}{3})}{2^{4/3}\Gamma(n+\frac{5}{3})\Gamma(\frac{2}{3}-m)}, \tag{16}$$

with $a_1 = 1.159595266963929$. Therefore, the first several terms are

$$(\cos\theta)^{1/3} = a_1\left[\cos\theta - \frac{\cos(3\theta)}{5} + \frac{\cos(5\theta)}{10} - \frac{7\cos(7\theta)}{110} + \frac{\cos(9\theta)}{22} - \frac{13\cos(11\theta)}{374} + \cdots\right]. \tag{17}$$

Substituting Eq. (16) into the right-hand side of Eq. (14) gives

$$\ddot{x}_1 + \Omega^2 x_1 = (\Omega^2 A - A^{1/3}a_1)\cos(\Omega t) - A^{1/3}\sum_{n=1}^{\infty} a_{2n+1}\cos[(2n+1)\Omega t]. \tag{18}$$