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On some operations and density of m-polar fuzzy graphs

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ABSTRACT

The theoretical concepts of graphs are highly utilized by computer science applications, social sciences, and medical sciences, especially in computer science for applications such as data mining, image segmentation, clustering, image capturing, and networking. Fuzzy graphs, bipolar fuzzy graphs and the recently developed m-polar fuzzy graphs are growing research topics because they are generalizations of graphs (crisp). In this paper, three new operations, i.e., direct product, semi-strong product and strong product, are defined on m-polar fuzzy graphs. It is proved that any of the products of m-polar fuzzy graphs are again an m-polar fuzzy graph. Sufficient conditions are established for each to be strong, and it is proved that the strong product of two complete m-polar fuzzy graphs is complete. If any of the products of two m-polar fuzzy graphs G_1 and G_2 are strong, then at least G_1 or G_2 must be strong. Moreover, the density of an m-polar fuzzy graph is defined, the notion of balanced m-polar fuzzy graph is studied, and necessary and sufficient conditions for the preceding products of two m-polar fuzzy balanced graphs to be balanced are established. Finally, the concept of product m-polar fuzzy graph is introduced, and it is shown that every product m-polar fuzzy graph is an m-polar fuzzy graph. Some operations, like union, direct product, and ring sum are defined to construct new product m-polar fuzzy graphs.

1. Introduction

Presently, science and technology is characterised by complex processes and phenomena for which complete information is not always available. For such cases, mathematical models are developed to handle various types of systems containing elements of uncertainty. A large number of these models is based on fuzzy sets. Graph theory has numerous applications to problems in computer science, electrical engineering, systems analysis, operations research, economics, networking routing, and transportation. Considering the fuzzy relations between fuzzy sets, Rosenfeld [18] introduced the concept of fuzzy graphs in 1975 and later developed the structure of fuzzy graphs. Mordeson and Nair [15] defined the complement of fuzzy graphs, which was further studied by Sunitha and Kumar [21]. The concepts of weak isomorphism, co-weak isomorphism and isomorphism between fuzzy graphs was introduced by Bhutani in Ref. [3]. Several researchers have worked on fuzzy graphs. Samanta and Pal introduced several types of fuzzy graphs, such as fuzzy planar graphs [29], fuzzy competition graphs [26,27], fuzzy tolerance graphs [22], and fuzzy threshold graphs [23]. Some more work on fuzzy graphs can be found on [4,12,16,17].

In 1994, Zhang [31–33] developed the concept of bipolar fuzzy sets as a generalization of fuzzy sets. The idea behind such description is connected with the existence of "bipolar information" (i.e., positive information and negative information) about the given set. Positive information represents what is granted to be possible, whereas negative information represents what is considered to be impossible. In 2011, using the concepts of bipolar fuzzy sets, Akram [1] introduced bipolar fuzzy graphs and defined different operations. Using this definition of bipolar fuzzy graphs, research is ongoing. Some work on bipolar fuzzy graphs may be found on [6,7,24,25,28,30]. Talebi and Rashmanlou [2] studied the complement and isomorphism of bipolar fuzzy graphs. Rashmanlou et al. [19,20] studied bipolar fuzzy graphs and bipolar fuzzy graphs with categorical properties.

In 2014, Juanjuan Chen et al. [5] introduced the notion of m-polar fuzzy sets as a generalization of bipolar fuzzy sets and showed that bipolar fuzzy sets and 2-polar fuzzy sets are cryptomorphic

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mathematical notions and that we can obtain one from the other. The idea behind this is that "multipolar information" (not just bipolar information, which corresponds to two-valued logic) exists because data of real world problems sometimes come from multiple agents. For example, the exact degree of telecommunication safety of mankind is a point in $[0,1]^n$ ($n \approx 7\times 10^9$) because different persons have been monitored different times. There are many other examples, such as truth degrees of a logic formula that are based on n logic implication operators ($n \geq 2$), similarity degrees of two logic formulas that are based on n logic implication operators ($n \geq 2$), ordering results of a magazine, ordering results of a university, and inclusion degrees (accuracy measures, rough measures, approximation qualities, fuzziness measures, and decision preformation evaluations) of a rough set.

Ghorai and Pal [8] introduced the notion of generalized m-polar fuzzy graphs as a generalization of bipolar fuzzy graphs and defined different operations. In Ref. [9], they studied the complement and isomorphism of m-polar fuzzy graphs. Because of the importance of m-polar fuzzy graphs mentioned in Refs. [8,9], we investigated m-polar fuzzy graphs. In this paper, three new operations are defined on the m-polar fuzzy graph, including direct product, semi-strong product and strong product. Any of the products of m-polar fuzzy graphs are again an m-polar fuzzy graph. Sufficient conditions are established for each one to be strong, and it is proved that the strong product of two complete m-polar fuzzy graphs is complete. If any of the products of two m-polar fuzzy graphs G_1 and G_2 are strong, then at least G_1 or G_2 must be strong. Moreover, the density of an m-polar fuzzy graph is defined, the notion of balanced m-polar fuzzy graphs is studied, and the necessary and sufficient conditions for the preceding products of two m-polar fuzzy balanced graphs to be balanced are established. Finally, the concept of product m-polar fuzzy graphs is introduced, and it is proved that every product m-polar fuzzy graph is an m-polar fuzzy graph. Some operations, like union, direct product, and ring sum, are defined to construct new product m-polar fuzzy graphs.

2. Preliminaries

In this section, we recall some definitions of fuzzy graphs, m-polar fuzzy sets, and m-polar fuzzy relations, which are defined below. For further study, see Refs. [5,8,9,11,13–15].

Definition 2.1. [15] *A fuzzy graph with V as the underlying set is a triplet $G = (V,\sigma,\mu)$, where $\sigma:V \to [0,1]$ is a fuzzy subset of V and $\mu:V\times V \to [0,1]$ is a fuzzy relation on σ, such that $\mu(x,y) \leq \sigma(x) \wedge \sigma(y)$ for all $x,y \in V$, where $\wedge$ stands for the minimum.*

The underlying crisp graph of G is denoted by $G^* = (\sigma^*,\mu^*)$, where $\sigma^* = \{x \in V:\sigma(x) > 0\}$ and $\mu^* = \{(x,y) \in V\times V:\mu(xy) > 0\}$.

Definition 2.2. [10] *A fuzzy graph $G = (V,\sigma,\mu)$ is complete if $\mu(x,y) = \sigma(x) \wedge \sigma(y)$ for all $x,y \in V$.*

The main purpose of this paper is to study m-polar fuzzy graphs based on m-polar fuzzy sets, which is defined below.

Throughout the paper, $[0,1]^m$ (m-power of $[0,1]$) is considered to be a poset with point-wise order $\leq$, where m is a natural number. $\leq$ is defined by $x \leq y \Leftrightarrow$ for each $i = 1,2,\ldots,m$; $p_i(x) \leq p_i(y)$ where $x,y \in [0,1]^m$ and $p_i:[0,1]^m \to [0,1]$ is the i-th projection mapping.

Definition 2.3. [5] *An m-polar fuzzy set (or a $[0,1]^m$-set) on X is a mapping $A:X \to [0,1]^m$. The set of all m-polar fuzzy sets on X is denoted by $m(X)$.*

Definition 2.4. [8] *Let A and B be two m-polar fuzzy sets in X. Then $A\cup B$ and $A\cap B$ are also m-polar fuzzy sets in X defined by:*

$p_i\circ(A\cup B((x) = \{p_i\circ A(x)\vee p_i\circ B(x)\}$ *and*

$p_i\circ(A\cap B)(x) = \{p_i\circ A(x)\wedge p_i\circ B(x)\}$ *for $i = 1,2,\ldots,m$ and $x \in X$* ($\vee$ *stands for maximum*).

$A\subseteq B$ if and only if for each $i = 1,2,\ldots,m$ and $x \in X$, $p_i\circ A(x) \leq p_i\circ B(x)$.

$A = B$ if and only if for each $i = 1,2,\ldots,m$ and $x \in X$, $p_i\circ A(x) = p_i\circ B(x)$.

Definition 2.5. [8] *Let A be an m-polar fuzzy set on a set X. An m-polar fuzzy relation on A is an m-polar fuzzy set B of $X\times X$ such that $B(x,y) \leq \min\{A(x),A(y)\}$ for all $x, y\in X$ i.e., for each $i = 1,2,\ldots,m$, for all $x,y \in X$, $p_i\circ B(x,y) \leq \min\{p_i\circ A(x),p_i\circ A(y)\}$.*

An m-polar fuzzy relation B on X is called symmetric if $B(x,y) = B(y,x)$ for all $x,y \in X$.

Definition 2.6. [10] *The semi-strong product of two fuzzy graphs $G_1 = (\sigma_1,\mu_1)$ and $G_2 = (\sigma_2,\mu_2)$ of the graphs $G_1^* = (V_1,E_1)$ and $G_2^* = (V_2,E_2)$, where it is assumed that $V_1\cap V_2 = \varnothing$, is defined to be the fuzzy graph $G_1\bullet G_2 = (\sigma_1\bullet\sigma_2,\mu_1\bullet\mu_2)$ of the graph $G^* = (V_1\times V_2,E)$* respectively, *such that $E = \{(u,v_1)(u,v_2)|u\in V_1, v_1v_2\in E_2\}\cup\{(u_1,v_1)(u_2,v_2)|u_1u_2\in E_1, v_1v_2\in E_2\}$,*

$(\sigma_1\bullet\sigma_2)(u,v) = \sigma_1(u)\wedge\sigma_2(v)$ *for all* $(u,v)\in V_1\times V_2$,

$(\mu_1\bullet\mu_2)((u,v_1)(u,v_2)) = \sigma_1(u)\wedge\mu_2(v_1v_2)$ *and*

$(\mu_1\bullet\mu_2)((u_1,v_1)(u_2,v_2)) = \mu_1(u_1u_2)\wedge\mu_2(v_1v_2)$.

Definition 2.7. [10] *The strong product of two fuzzy graphs $G_1 = (\sigma_1,\mu_1)$ and $G_2 = (\sigma_2,\mu_2)$ of the graphs $G_1^* = (V_1,E_1)$ and $G_2^* = (V_2,E_2)$* respectively, *where it is assumed that $V_1\cap V_2 = \varnothing$, is defined to be the fuzzy graph $G_1\otimes G_2 = (\sigma_1\otimes\sigma_2,\mu_1\otimes\mu_2)$ of the graph $G^* = (V_1\times V_2,E)$, such that $E = \{(u,v_1)(u,v_2)|u\in V_1, v_1v_2\in E_2\}\cup\{(u_1,w)(u_2,w)|w\in V_2, u_1u_2\in E_1\}\cup\{(u_1,v_1)(u_2,v_2)|u_1u_2\in E_1, v_1v_2\in E_2\}$,*

$(\sigma_1\otimes\sigma_2)(u,v) = \sigma_1(u)\wedge\sigma_2(v)$ *for all* $(u,v) \in V_1\times V_2$,

$(\mu_1\otimes\mu_2)((u,v_1)(u,v_2)) = \sigma_1(u)\wedge\mu_2(v_1v_2)$,

$(\mu_1\otimes\mu_2)((u_1,w)(u_2,w)) = \sigma_2(w)\wedge\mu_1(u_1u_2)$ *and*

$(\mu_1\otimes\mu_2)((u_1,v_1)(u_2,v_2)) = \mu_1(u_1u_2)\wedge\mu_2(v_1v_2)$.

Definition 2.8. [10] *The direct product of two fuzzy graphs $G_1 = (\sigma_1,\mu_1)$ and $G_2 = (\sigma_2,\mu_2)$ of the graphs $G_1^* = (V_1,E_1)$ and $G_2^* = (V_2,E_2)$* respectively, *such that $V_1\cap V_2 = \varnothing$, is defined to be the fuzzy graph $G_1\sqcap G_2 = (\sigma_1\sqcap\sigma_2,\mu_1\sqcap\mu_2)$ of the graph $G^* = (V_1\times V_2,E)$, such that $E = \{(u_1,v_1)(u_2,v_2)|u_1u_2\in E_1, v_1v_2\in E_2\}$,*

$(\sigma_1\sqcap\sigma_2)(u,v) = \sigma_1(u)\wedge\sigma_2(v)$ *for all* $(u,v) \in V_1\times V_2$, *and*

$(\mu_1\sqcap\mu_2)((u_1,v_1)(u_2,v_2)) = \mu_1(u_1u_2)\wedge\mu_2(v_1v_2)$.

For a given set V, define an equivalence relation on $V\times V - \{(x,x) : x\in V\}$ as follows:

$(x_1,y_1) \sim (x_2,y_2) \Leftrightarrow$ *either* $(x_1,y_1) = (x_2,y_2)$ *or* $x_1 = y_2$ *and* $y_1 = x_2$.

The quotient set obtained in this way is denoted by $\widetilde{V^2}$, and the equivalence class that contains the element (x,y) is denoted as xy or yx.

Throughout this paper, G^* represents a crisp graph, and G is an m-polar fuzzy graph of G^*.

3. *m*-polar fuzzy graphs

In this section, we briefly recall some basic definitions related to m-polar fuzzy graphs.

Definition 3.1. [8] *An m-polar fuzzy graph of a graph $G^* = (V,E)$ is a pair $G = (A,B)$ where $A:V \to [0,1]^m$ is an m-polar fuzzy set in V and $B:\widetilde{V^2}\to[0,1]^m$ is an m-polar fuzzy set in $\widetilde{V^2}$, such that for each $i = 1,2,\ldots,m$; $p_i\circ B(xy) \leq \min\{p_i\circ A(x), p_i\circ A(y)\}$ for all $xy\in\widetilde{V^2}$ and $B(xy) = 0$ for all $xy\in\widetilde{V^2} - E$, $(0 = (0,0,\ldots,0)$ is the smallest element in $[0,1]^m)$.*

A is called the m-polar fuzzy vertex set of G, and B is called the m-polar fuzzy edge set of G.