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Direct calculation of wind turbine tip loss

D.H. Wood [a, *], V.L. Okulov [b, c], D. Bhattacharjee [a, 1]

[a] Department of Mechanical and Manufacturing Engineering, University of Calgary, Calgary, T2N 1N4, AB, Canada
[b] Department of Wind Energy, Technical University of Denmark, DK 2800, Lyngby, Denmark
[c] Institute of Thermophysics, Siberian Branch of the Russian Academy of Sciences, 630090, Novosibirsk, Russia

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ABSTRACT

The usual method to account for a finite number of blades in blade element calculations of wind turbine performance is through a tip loss factor. Most analyses use the tip loss approximation due to Prandtl which is easily and cheaply calculated but is known to be inaccurate at low tip speed ratio. We develop three methods for the direct calculation of the tip loss. The first is the computationally expensive calculation of the velocities induced by the helicoidal wake which requires the evaluation of infinite sums of products of Bessel functions. The second uses the asymptotic evaluation of those sums by Kawada. The third uses the approximation due to Okulov which avoids the sums altogether. These methods are compared to the tip loss determined independently and exactly for an ideal three-bladed rotor at tip speed ratios between zero and 15. Kawada's asymptotic approximation and Okulov's equations are preferable to the Prandtl factor at all tip speed ratios, with the Okulov equations being generally more accurate. In particular the tip loss factor exceeds unity near the axis of rotation by a large amount at all tip speed ratios, which Prandtl's factor cannot reproduce. Neither the Kawada nor the Okulov equations impose a large computational burden on a blade element program.

1. Introduction

The tip loss factor, F, is used routinely in blade element and other analyses of wind turbine rotors to account for the finite number of blades, N. It reduces the calculated rotor power by 5–10%, eg Clifton-Smith [1] and, therefore, must be represented accurately. Blade element momentum theory (BEMT) balances the forces on the blades against the changes in momentum and angular momentum in the annular streamtube that intersects the element, eg Hansen [2]. The streamtube equations are formulated in terms of the average velocity in the tube whereas the forces on the blades depend on the velocity at the blade. These velocities become equal only when $N \to \infty$. In terms of the axial induction factor, a, the axial velocity normalized by the wind speed is $1 - a$ in the streamtube and $1 - a_b$ at the blade. Following Glauert [3], most authors, including Shen et al. [4], have analyzed the tip loss factor as $F = a/a_b$. Refs [1] and [4] discussed the incorporation of F in blade element equations. The latter recommends the method of de Vries [5] as giving the most consistent results. The purpose of this paper is *not* to continue that discussion but to investigate the determination of F.

Nearly all BEMT analyses use Prandtl's approximation for F, eg Ref. [3] and Wald [6], which will be denoted F_P. Its usual form is

$$F_P = \frac{2}{\pi} cos^{-1}\left[exp\left(-\frac{N(1-r)}{2rsin\phi}\right)\right] \quad (1)$$

where r is the radius normalized by the blade tip radius, and ϕ is the "inflow" angle between the total velocity at the blade and the axial direction. Note that $F_P \le 1$. F_P has the great advantage of being easily and quickly determined in BEMT computer codes that must iterate to convergence at each blade element. Prandtl derived the original form of F_P by representing the helicoidal trailing vorticity of a blade by semi-infinite two-dimensional laminae as explained by Wald [6]. This restricts F_P to high tip speed ratio, λ, whereas there are important examples of wind turbine operation at low λ, for example, waterpumping windmills and conventional large turbines operating near the shut down wind speed. Furthermore, there are technologies, such as diffuser-augmented turbines, for

* Corresponding author.
E-mail address: dhwood@ucalgary.ca (D.H. Wood).
[1] Current address: Department of Mechanical Engineering, University of Alberta, Edmonton, T6G 1H9, AB, Canada.

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which F_P is unlikely to provide an accurate tip loss.

Shen et al. [4] adapted (1) to give "a correction to the two-dimensional aerofoil data in the tip region" by multiplying the argument of the exponential by

$$g = exp[-c_1(N\lambda - c_2) + c_3] \tag{2}$$

with provisional values of $c_1 = 0.125$, $c_2 = 21$, and $c_3 = 0.1$.

Both a and a_b are induced by the trailing vortices only, at least to the accuracy of the lifting line representation of a rotor and its wake which is implicitly assumed in BEMT. Thus it is possible to determine F directly, but this has never been done, presumably because of the high computational cost. This paper presents three methods for calculating F in BEMT and tests them over the range $0 < \lambda < 15$ for an ideal three-bladed rotor. Comparison is made against the exact F determined independently. The first method evaluates the equations for the velocity induced by the helical vortices in the wake. This is computationally too expensive to be used routinely, but sets the standard for the others, which are approximate methods. The second is the asymptotic equations of Kawada [7] and the third is the equations of Okulov [8]. All methods are more accurate than F_P at all investigated tip speed ratios. Moreover, the second and third do not significantly increase the computational burden compared to the use of Equation (1).

We test the accuracy of tip loss calculations for a Betz-Goldstein optimized rotor. The Goldstein function gives the optimal loading on an $N-$ bladed rotor according to

$$G_N(r) = N\Gamma_N(r)\lambda/(4\pi w(1-w)) \tag{3}$$

where w is the constant axial velocity of the rigid helicoidal sheet representing the wake at the rotor, as detailed by Wald [6] and Okulov & Sørensen [9]. The implication is that vortex pitch, p, given by

$$p = dz/d\theta = (1-w)/\lambda \tag{4}$$

where z is axial location of a point on the vortex and θ is its circumferential position, is constant with radius. The next Section formulates the three direct calculations of F. Section 3 describes the implementation of those calculations for optimal rotors and explains the determination of w. Section 4 presents the results for a wide range of λ and the final section gives the conclusions. The Appendix gives further details of the behaviour of F for small radius at low λ.

2. The tip loss due to a helical wake

One of the reasons that F is not computed directly is the complexity of the equations describing the velocities induced by the helicoidal vortex sheet in the wake of a wind turbine rotor, Okulov et al. [10]. Since [10] was published, we became aware of the ground-breaking work of Kawada [11] [7], (see Fukumoto et al. [12]) which will be used extensively in the present analysis. There are no known analytical solutions for the velocities induced by expanding helical vortices, so we begin by stating the equation for the velocity induced at radius r by a semi-infinite vortex originating at the same z but radius t and pitch p. t and p are constant throughout the wake. The combination of all such vortices in the wake is sufficient to determine a and a_b, because the bound vorticity of the blades cannot contribute to the average induced velocity through the rotor or to the velocity at the blade. Kawada [7] analyzed the induced azimuthal velocity and we follow this lead. For the Betz-Goldstein rotors studied here, it does not matter which velocity component is used to find F because the vortex pitch is constant across the wake. The equation for the induced azimuthal velocity at the blade $u_{\theta,b}$ is

$$\begin{aligned} u_{\theta,b}(r) &= \frac{N\Gamma t}{2\pi p r}S_1 && \text{for } r < t \text{ and} \\ &= \frac{N\Gamma}{4\pi r} + \frac{N\Gamma t}{2\pi p r}S_3 = u_\theta(r) + \frac{N\Gamma t}{2\pi p r}S_3 && \text{for } r > t \end{aligned} \tag{5}$$

using the notation of Hardin [13]. The circulation, Γ, and the lengths are normalized by the wind speed and tip radius. The tip loss is given by $F = u_\theta(r)/u_{\theta,b}(r)$ where u_θ is the average velocity in the streamtube. In (5)

$$S_1 = \sum_1^\infty mK'_m(mt/p)I_m(mr/p)cos(m\theta) \tag{6}$$

and

$$S_3 = \sum_1^\infty mI'_m(mt/p)K_m(mr/p)cos(m\theta) \tag{7}$$

where $I(.)$ and $K(.)$ are modified Bessel functions in standard notation and the differentiation is with respect to the argument. The azimuthal angle θ is zero for a vortex originating from the same blade. Equations (5)–(7) are due to Hardin [13] but Kawada [11] derived analogous equations for the light-loading approximation introduced by Goldstein [14].

In the equation for F, u_θ is easy to evaluate, but the calculation of $u_{\theta,b}$ using (5)is computationally unattractive because of the infinite sums in (6) and (7) and the need to express each of the two derivatives as Bessel functions of adjacent orders. Nevertheless Equations (5)–(7) for N trailing vortices can be written in a form suitable for BEMT:

$$\begin{aligned} u_{\theta,b}(r) = u_\theta(r) + \frac{1}{2\pi p r} \Bigg(& \sum_{m=1}^{\infty} mNK_{mN}(mNr/p)\int_0^r t\frac{\partial\Gamma}{\partial t}(I_{mN-1}(mNt/p) \\ & + I_{mN+1}(mNt/p))dt - \sum_{m=1}^{\infty} mNI_{mN}(mNr/p) \\ & \int_r^1 t\frac{\partial\Gamma}{\partial t}(K_{mN-1}(mNt/p) + K_{mN+1}(mNt/p))dt \Bigg) \end{aligned} \tag{8}$$

Equation (8) will be approximated to find a_b at the midpoint of each element from the contributions of the distinct vortices originating at the junctions of all the elements with a strength equal to the difference in the circulation of the adjacent elements. This formulation avoids the self-induced velocity of the trailing vortices which depends on the singularities due to the swirl and the vortex curvature, Boersma & Wood [15]. The summations in (8) are written separately to correspond with the two equations in (5) and because the two integrals were evaluated separately.

In obtaining Equation (8) the expressions for S_1 and S_3 have been summed over the N vortices at each t. This results in a major simplification because

$$\begin{aligned} \sum_{j=0}^{N-1} cos(2\pi mj/N) &= N \quad \text{if } m \text{ is a multiple of } N \text{ and} \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{9}$$

As first realized by Kawada (7), only terms that are multiples of