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Symmetric matrix-valued transmitting boundary formulation in the time-domain for soil–structure interaction problems

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ABSTRACT

Time-domain formulations for soil–structure interaction problems with an unbounded soil-domain (the so-called *far-field*) including wave propagation require such time-domain formulations for both parts, soil and structure. For the structure (the *near-field*), typically treated by a finite element approach, the time-domain is used from the very beginning of the procedure. However, for the unbounded soil a representation by means of a frequency-dependent dynamic stiffness is usually available and it becomes necessary to devise techniques for switching from the frequency- to the time-domain.

For various special cases in solid mechanics (e.g. plane, cylindrical and spherical waves) one-dimensional formulations in space have been used to derive scalar dynamical stiffness, to establish corresponding rational functions in the frequency-domain and transfer them into the time-domain in order to couple the near- and the far-field.

A complete three-dimensional analysis for pile-groups through a linear homogeneous unbounded soil-domain and the corresponding description in the time-domain have already been treated by Cazzani and Ruge (2012, 2013) by means of a fully matrix-valued rational representation of a set of dynamic stiffness matrices $\mathbf{K}(\Omega)$, as a function of the angular frequency Ω. However, the symmetry of the input stiffness $\mathbf{K}(\Omega)$ has not been maintained for the corresponding representation in the time-domain.

This paper presents a fully matrix-valued rational formulation which does transfer the symmetry of $\mathbf{K}(\Omega)$ to the corresponding formulation in the time-domain. Thus, the numerical treatment of the whole soil–structure interaction problem, coupling the far-field and near-field systems, can take advantage of algorithms for symmetric algebraic problems.

1. Introduction

The modeling of transient processes in the time-domain for unbounded space-domains like soil including a consistent description of wave-propagation is still a challenge, though it has been the subject of research for more than 30 years. A rather comprehensive report on the state of the art up to year 2011 has been presented by Birk [1]. A recently published paper [2] by the authors contains a selection of some typical and essential contributions.

For bounded fields in the space domain the finite element method is well-established in Computational Engineering. For unbounded domains the finite concept has been further developed towards infinite elements with special frequency-dependent shape functions [3,4] which establish the so-called *transient wave envelope* concept including an inverse Fourier transform. This procedure is rather popular in acoustics.

Whereas today in Structural Engineering especially two methods modeling the unbounded domain are used for soil–structure interaction problems: the Boundary Element Method [5] in a symmetric version [6,7] and the Scaled Boundary Element method [8] with several improvements; some of them are mentioned in [9]. Both methods end up with frequency dependent symmetric dynamic stiffness matrices and suitable procedures have to be developed to change over to the time-domain. One approach, which is shown in detail in [10], requires the associated unit-impulse response matrix by means of an inverse Fourier transformation, followed by the evaluation of a convolution integral. However, this process is temporally nonlocal and, thus, numerically complicated.

A temporally local procedure for coupling an unbounded soil domain with a bounded structure modeled by classical finite elements in the time-domain starts with a boundary condition $\mathbf{f}_c = \mathbf{K}_c(\Omega)\mathbf{u}_c$ in the interface with a pair of nodal interface-forces $(\mathbf{f}_c; -\mathbf{f}_c)$ and the interface-displacements $\mathbf{u}_c$. The dynamic stiffness

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matrix $\mathbf{K}_c(\Omega)$, sometimes called DtN-maps [11], depends on the angular frequency Ω. To describe the whole coupled problem in the time-domain and to find solutions for arbitrary excitations, the frequency-domain DtN-mapping has to be cast into the time-domain.

This task has been elaborated and introduced by Wolf [12] for the treatment of soil–structure interaction problems by replacing a set of scalar dynamic stiffness coefficients by a frequency-dependent rational function $f_c = (P/Q)u_c$ with scalar polynomials $P(\Omega)$ and $Q(\Omega)$, which can be reduced to a linear system in the frequency-domain and thus to a system of ordinary differential equations (ODEs): $\mathbf{A}\dot{\mathbf{z}} - \mathbf{B}\mathbf{z} = \mathbf{0}$ in the time-domain. In system-theory, this is a classical, well established approach, and is described in textbooks like [13].

However, for dynamic stiffness matrices with several physical degrees-of-freedom (DOFs) in the interface between the unbounded and the bounded domain, matrices **A** and **B** describing the system of ODEs become unsymmetric for a full matrix-valued rational approximation $\mathbf{Q}(\Omega)\mathbf{f}_c = \mathbf{P}(\Omega)\mathbf{u}_c$, with the corresponding dynamic stiffness matrix $\mathbf{K}_c(\Omega) = \mathbf{Q}^{-1}\mathbf{P}$ as it has been shown in some previous papers like [14,15].

Here **P** and **Q** are matrix-valued polynomials depending on the angular frequency Ω. If the interface between structure and soil is given by a rigid foundation-plate, $\mathbf{K}_c(\Omega)$ is a 6×6 matrix corresponding to three translational and three rotational DOFs.

Thus, when coupling the two fields, soil and structure, the whole algebraic representation becomes a non-symmetric one: there is no more benefit from the symmetry related to the finite element model of the structure which typically dominates the problem by its hundreds of DOFs.

However, symmetry is a fundamental quality in Natural- and Engineering Sciences and if systems are characterized by symmetry their algebraic representation should reflect this property by means of symmetric matrices. Compared with non-symmetric matrices, they need only roughly half of the storage, the numerical tools for solving symmetric algebraic equations are more robust and the amount of arithmetic operations is much less.

Thus preserving symmetry — if there is any — is of the utmost importance for an effective formulation of a physical problem.

A compromise in describing the DtN-maps by using only a scalar formulation $Q(\Omega)$ instead of the matrix-valued one, $\mathbf{Q}(\Omega)$, for the force-side, $Q(\Omega)\mathbf{f}_c = \mathbf{P}(\Omega)\mathbf{u}_c$ with the corresponding stiffness matrix

$$\mathbf{K}_c(\Omega) = Q^{-1}\mathbf{P} = \mathbf{P}Q^{-1} = \frac{\mathbf{P}}{Q}$$

gives unequal weights to the nodal quantities $\mathbf{f}_c$ and $\mathbf{u}_c$ (which, in principle, have equal importance) but, on the other hand, results in symmetric system matrices **A** and **B** [16].

Here a symmetric representation in the time-domain will be presented which includes matrix-valued operators for both mechanical quantities; nodal forces and displacements.

The paper is organized as follows: in Section 2 the standard formulation of a typical multi-DOFs structural dynamics problem which is ruled by a system of second-order ODEs is considered, and it is shown that it can be transformed, by preserving symmetry of governing matrices, into a system of first-order ODEs of twice its size. In Section 3, the soil–structure coupled problem is considered and the frequency-to-time transformation, as it has been elaborated in [14,15] will be recalled, showing its essential steps. Then the rational representation $\mathbf{K}(\Omega) = \mathbf{Q}^{-1}\mathbf{P}$ used in the previous papers is combined with the conjugate formulation $\mathbf{K}(\Omega) = \tilde{\mathbf{P}}\tilde{\mathbf{Q}}^{-1}$.

Combining the conjugate pair $\mathbf{K}(\Omega) = \mathbf{Q}^{-1}\mathbf{P}$ and $\mathbf{K}(\Omega) = \tilde{\mathbf{P}}\tilde{\mathbf{Q}}^{-1}$ together in one common state equation results in symmetric state matrices **A** and **B**, as it will be shown in Section 4.

In Section 5 the new symmetric representation is used to solve a soil–rotor foundation interaction problem with a transient excitation which has been already treated by the non-symmetric formulation in [17]. The results are compared and several aspects concerning the sensitivity of the linear algebraic solvers used and their accuracy in combination with the machine precision are discussed in detail.

2. Alternate formulations of the structural dynamics problem

As it has been mentioned already, the DtN-maps $\mathbf{f}_c = \mathbf{K}_c(\Omega)\mathbf{u}_c$ in the coupling interface can be replaced by a system of ODEs in the time-domain $\mathbf{A}\dot{\mathbf{z}} - \mathbf{B}\mathbf{z} = \mathbf{r}_c$, where the coupling force $\mathbf{f}_c$ is a part of the right-hand side $\mathbf{r}_c$. In order to organize the coupling of this system of first order ODEs with the finite element model of the bounded structure it is advantageous to formulate its equation of motion as a set of first order ODEs, too.

In structural dynamics the standard form of the governing equation for system described by m physical DOFs is:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{D}\dot{\mathbf{x}} + \mathbf{C}\mathbf{x} = \mathbf{F}(t), \tag{1}$$

i.e. a set of m second-order ODEs where **M**, **D**, **C** are respectively the mass, damping and elastic stiffness matrices (all of them being symmetric ones), while $\mathbf{x}$, $\dot{\mathbf{x}}$, $\ddot{\mathbf{x}}$, **F** are respectively (possibly generalized) displacements, velocities, accelerations and time-dependent external forces corresponding to the physical DOFs.

This form (1) is suitable for modal analysis and even for time integration provided that matrix **D** can be diagonalized by the *same* transformation which diagonalizes both **M** and **C**: in such a case a system of m decoupled second order ODEs in terms of principal coordinates is obtained and the solution, at least numerically, if **F** exhibits a complicated time dependence, can be easily computed.

It can be shown that this requirement is satisfied when **D** is proportional to **M** (*mass damping*) or to **C** (*stiffness damping*) or to a combination of both; alternatively **D** might be defined as a diagonal matrix in the space of principal coordinates (*modal damping*) and then back transformed into the space of physical DOFs.

In all other cases it is more useful switching from (1) to a formulation where the governing equation is expressed as a $2m$ system of first-order ODEs: this is easily accomplished by introducing a new set of variables, **y**, defined as

$$\mathbf{y} - \dot{\mathbf{x}} = \mathbf{0}, \tag{2}$$

where **0** is a null vector having m rows.

By introducing (2) — which corresponds to assuming velocities as independent variables — into (1) and rearranging the terms the following two alternate forms can be obtained

$$\begin{bmatrix} \mathbf{M} & \mathbf{O} \\ \mathbf{O} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{y}} \\ \dot{\mathbf{x}} \end{bmatrix} + \begin{bmatrix} \mathbf{D} & \mathbf{C} \\ \mathbf{I} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{0} \end{bmatrix}; \tag{3}$$

$$\begin{bmatrix} \mathbf{M} & \mathbf{D} \\ \mathbf{O} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{y}} \\ \dot{\mathbf{x}} \end{bmatrix} + \begin{bmatrix} \mathbf{O} & \mathbf{C} \\ \mathbf{I} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{0} \end{bmatrix}, \tag{4}$$

where **O** and **I** are, respectively, a square $m \times m$ null matrix and the m-th order identity matrix. Of these two forms the latter one (4) cannot be easily put into a symmetric form, while this is possible for the former one (3) provided that the lower partition is pre-multiplied by **C**:

$$\begin{bmatrix} \mathbf{M} & \mathbf{O} \\ \mathbf{O} & -\mathbf{C} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{y}} \\ \dot{\mathbf{x}} \end{bmatrix} + \begin{bmatrix} \mathbf{D} & \mathbf{C} \\ \mathbf{C} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{0} \end{bmatrix}. \tag{5}$$

Moreover, if **z** denotes a $2m$ vector collecting the state variables $(\mathbf{y}, \mathbf{x})$, i.e. $\mathbf{z}^T = [\mathbf{y}^T \ \mathbf{x}^T]$, and, similarly, external forces are written as $\mathbf{f}^T = [\mathbf{F}^T \ \mathbf{0}^T]$, then (5) can be synthetically written as