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A note on plane-wave approximation

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ABSTRACT

It is shown that the plane-wave assumption for incident SH waves is a good approximation for cylindrical waves radiated from a finite source even when it is as close as twice the size of inhomogeneity. It is concluded that for out-of-plane SH waves the plane-wave approximation should be adequate for many earthquake engineering studies.

1. Introduction

Many earthquake engineering studies of amplification of incident seismic waves by internal inhomogeneities and by surface topography assume excitation by plane harmonic waves [1–18]. It is assumed in these studies that when the spherical and cylindrical wave fronts are sufficiently far from the earthquake source the plane-wave approximation may represent an adequate approximation. Most studies assume periodic excitation and present the results in terms of transfer-function amplitudes, usually along the ground surface and in the vicinity of inhomogeneity. The significance of these studies has been (1) in showing how the two- and three-dimensional interference, focusing, scattering, and diffraction of linear plane waves by inhomogeneities lead to changes in the amplitudes, frequencies, and locations of the observed peaks of transfer functions; and (2) in comparing the relative significance of surface topography and interior material inhomogeneities (sedimentary valleys) [18]. A review of these studies is presented in [11].

The purpose of this brief note is to show, by using elementary examples of SH waves, that the plane-wave approximation does indeed provide reasonable and useful approximation. We will show this by comparing the transfer functions for incident plane waves with the transfer functions for excitation by cylindrical waves emanating from a periodic finite source of SH waves.

2. Model

The model we consider consists of a semi-circular sedimentary valley, with radius a, surrounded by the elastic homogeneous and isotropic half-space (Fig. 1). The half-space is characterized by densityρ_s and shear-wave velocity c_s, while the semi-cylindrical valley is described by ρ_v and c_v. The fault, which radiates periodic SH waves, is located at $r=a_f$, between the angles $\pi+\alpha_f-\alpha_{fl}/2$ and $\pi+\alpha_f+\alpha_{fl}/2$. The fault width is $a_f\alpha_{fl}$.

3. Solution

To describe radiation from the fault, two displacement fields are defined inside the half space: $u_s=u_{s1}$ for $a<r<a_f$ and $u_s=u_{s2}$ for $a_f<r<\infty$. u_{s1} contains two displacement fields, u_{s1c} represents cylindrical waves propagating toward $r=0$, and u_{s1g} represents reflected waves from the valley so that in that region $u_{s1}=u_{s1g}+u_{s1c}$. u_{s2} represents the waves propagating away from the origin $r=0$, and the waves inside the valley are u_v.

The governing equation for out-of-plane SH waves that are valid in both regions is

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)U(r,\theta,t)=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}U(r,\theta,t). \tag{1}$$

The time dependence of the solution will be taken as harmonic so that

$$U(r,\theta,t)=u(r,\theta)e^{-i\omega t}, \tag{2}$$

where ω is the angular frequency. When Eq. (2) is substituted into Eq. (1), there follows

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)u(r,\theta)e^{-i\omega t}=-\frac{\omega^2}{c^2}u(r,\theta)e^{-i\omega t}. \tag{3}$$

Next, we introduce the wave number, $k=\omega/c$, which gives

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+k^2\right)u(r,\theta)=0. \tag{4}$$

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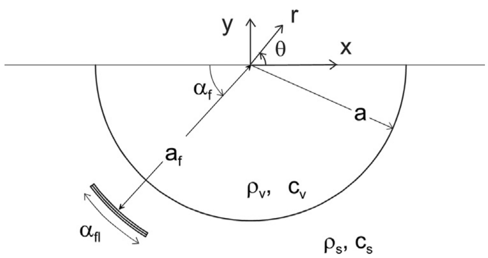

Fig. 1. Geometry of the problem.

When we assume that separation of the variables solves the problem $u(r,\theta)=R(r)\Theta(\theta)$, there follows

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+k^2\right)R(r)\Theta(\theta)=0 \tag{5a}$$

$$\Theta(\theta)\frac{\partial^2 R(r)}{\partial r^2}+\Theta(\theta)\frac{1}{r}\frac{\partial R(r)}{\partial r}+R(r)\frac{1}{r^2}\frac{\partial^2\Theta(\theta)}{\partial\theta^2}+R(r)\Theta(\theta)k^2=0 \tag{5b}$$

$$\frac{r^2}{R(r)}\frac{\partial^2 R(r)}{\partial r^2}+\frac{r}{R(r)}\frac{\partial R(r)}{\partial r}+\frac{1}{\Theta(\theta)}\frac{\partial^2\Theta(\theta)}{\partial\theta^2}+r^2k^2=0 \tag{5c}$$

$$\frac{r^2}{R(r)}\frac{\partial^2 R(r)}{\partial r^2}+\frac{r}{R(r)}\frac{\partial R(r)}{\partial r}+r^2k^2=-\frac{1}{\Theta(\theta)}\frac{\partial^2\Theta(\theta)}{\partial\theta^2}. \tag{5d}$$

This equation holds if both sides are equal to a constant. If this constant is chosen to be n^2, then

$$\frac{r^2}{R(r)}\frac{\partial^2 R(r)}{\partial r^2}+\frac{r}{R(r)}\frac{\partial R(r)}{\partial r}+r^2k^2=-\frac{1}{\Theta(\theta)}\frac{\partial^2\Theta(\theta)}{\partial\theta^2}=n^2 \tag{5e}$$

$$\frac{\partial^2\Theta(\theta)}{\partial\theta^2}+\Theta(\theta)n^2=0=>\Theta(\theta)=\beta e^{in\theta}+\beta e^{-in\theta} \tag{5f}$$

$$r^2\frac{\partial^2 R(r)}{\partial r^2}+r\frac{\partial R(r)}{\partial r}+R(r)(r^2k^2-n^2)=0, \tag{5g}$$

and after variable transformation, $\xi=rk$,

$$\frac{\partial R(r)}{\partial r}=\frac{\partial R(r)}{\partial\xi}\frac{\partial\xi}{\partial r}=\frac{\partial R(r)}{\partial\xi}k \tag{6a}$$

$$\frac{\partial}{\partial r}\left(\frac{\partial R(r)}{\partial r}\right)=\frac{\partial}{\partial r}\left(\frac{\partial R(r)}{\partial\xi}k\right)=\frac{\partial}{\partial\xi}\left(\frac{\partial R(r)}{\partial\xi}\right)\frac{\partial\xi}{\partial r}k=\frac{\partial^2 R(r)}{\partial\xi^2}k^2 \tag{6b}$$

$$r^2\frac{\partial^2 R(r)}{\partial\xi^2}k^2+r\frac{\partial R(r)}{\partial\xi}k+R(r)(r^2k^2-n^2)=0 \tag{6c}$$

$$\xi^2\frac{\partial^2 R(\xi/k)}{\partial\xi^2}+\xi\frac{\partial R(\xi/k)}{\partial\xi}+R(\xi/k)(\xi^2-n^2)=0. \tag{6d}$$

This is a Bessel differential equation, and its solution is

$$R(\xi/k)=C_n(\xi). \tag{7}$$

Changing the variables again, $R(r)=C_n(kr)$, the solution function will be $C_n(kr)e^{in\theta}$. Since the solution has to be periodic in θ, n has to be an integer. The solution is valid for all integer values of n, from minus infinity to plus infinity. Therefore, the general solution is a linear combination, as follows:

$$u(r,\theta)=\sum_{n=-\infty}^{\infty}A_nC_n(kr)e^{in\theta}. \tag{8}$$

here, A_n are complex constants to be determined by boundary conditions. C_n is a Bessel function with order n, which can be either J_n,Y_n, ${H_n}^{(1)}$, or ${H_n}^{(2)}$ depending on the physical conditions of the problem. ${H_n}^{(1)}$ describes the outgoing waves, while ${H_n}^{(2)}$ describes the incoming waves. Because the series above is convergent, it is possible to truncate it into a finite sum with N terms, as follows:

$$u(r,\theta)=\sum_{n=-N}^{N}A_nC_n(kr)e^{in\theta}. \tag{9}$$

The solutions for each sub-space in Fig. 1 are written below in such a way that the solutions that do not satisfy the Sommerfeld radiation condition are omitted. The displacement fields are

$$u_v=\sum_{n=-N}^{N}A_{v,n}J_n(k_{\beta v}r)e^{in\theta} \tag{10a}$$

$$u_{s1g}=\sum_{n=-N}^{N}A_{s1g,n}{H_n}^{(1)}(k_{\beta s}r)e^{in\theta} \tag{10b}$$

$$u_{s1c}=\sum_{n=-N}^{N}A_{s1c,n}{H_n}^{(2)}(k_{\beta s}r)e^{in\theta} \tag{10c}$$

$$u_{s1}=u_{s1g}+u_{s1c} \tag{10d}$$

$$u_{s2}=\sum_{n=-N}^{N}A_{s2,n}{H_n}^{(1)}(k_{\beta s}r)e^{in\theta}, \tag{10e}$$

where $k_{\beta s}$ and $k_{\beta v}$are wave numbers in the half-space and in the valley, respectively. In terms of η, these wave numbers will be $k_{\beta s}=\eta\pi/a$, and $k_{\beta v}=(k_{\beta s}c_s)/c_v$. η is the ratio of valley width over wavelength in the half-space ($\eta=2a/\lambda_s$). Lamé's second parameters (shear moduli) for half-space and valley are $\mu_s={c_s}^2\rho_s$ and $\mu_v={c_v}^2\rho_v$, respectively. The angular frequency is $\omega=k_{\beta s}c_s=k_{\beta v}c_v$.

For convenience, we introduce $\alpha_1=\pi+\alpha_f-\alpha_{fl}/2$, $\alpha_2=\pi+\alpha_f+\alpha_{fl}/2$ as new variables.

The boundary conditions are as follows:

Zero stress on a flat surface is

$$\frac{\mu_v}{r}\frac{\partial}{\partial\theta}u_v\bigg|_{\substack{\theta=0\\ \theta=\pi}}=\frac{\mu_s}{r}\frac{\partial}{\partial\theta}u_{s1}\bigg|_{\substack{\theta=0\\ \theta=\pi}}=\frac{\mu_s}{r}\frac{\partial}{\partial\theta}u_{s2}\bigg|_{\substack{\theta=0\\ \theta=\pi}}=0. \tag{11a}$$

The continuity of stress and displacement on the interface between the valley and the half-space is

$$\mu_v\frac{\partial}{\partial r}u_v\Big|_{r=a}=\mu_s\frac{\partial}{\partial r}u_{s1}\Big|_{r=a} \tag{11b}$$

$$u_v|_{r=a}=u_{s1}|_{r=a}. \tag{11c}$$

The continuity of stress in divided regions of half-space is

$$\mu_s\frac{\partial}{\partial r}u_{s1}\Big|_{r=a_f}=\mu_s\frac{\partial}{\partial r}u_{s2}\Big|_{r=a_f} \tag{11d}$$

The displacement difference in divided regions of half-space is

$$u_{s1}|_{r=a_f}-u_{s2}|_{r=a_f}=f(\theta). \tag{11e}$$

Where $f(\theta)$ is a function that satisfies the relative displacement difference (on the fault surface) between angles α_1 and α_2 (here assumed to be a constant), and the continuity of displacement elsewhere.

To satisfy the zero-stress condition on a flat surface, we introduce another fault, symmetric with regard to the x axis—that is, we employ the imaging method. With this imaginary fault, the $\bar{f}(\theta)$ function will take the following form:

$$\bar{f}(\theta)=\{H[\theta-\alpha_1]-H[\theta-\alpha_2]\}+\{H[\theta-(2\pi-\alpha_2)]-H[\theta-(2\pi-\alpha_1)]\} \tag{12}$$

and its finite Fourier transform becomes

$$f_n=\frac{1}{2\pi}\int_0^{2\pi}\bar{f}(\theta)e^{-in\theta}d\theta \tag{13a}$$