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Lumped-parameter model of foundations based on complex Chebyshev polynomial fraction

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A B S T R A C T

This paper uses the complex Chebyshev polynomials to develop the lumped-parameter (L-P) model of foundations. The method is an important extension to the approach adopting polynomials to model the dynamic properties of the foundation. Using the complex Chebyshev polynomials can reduce the unexpected wiggling in the foundation modeling, which inevitably occurs if using the simple polynomials of high degree. In the present analysis, the normalized flexibility function of foundation is expressed in terms of complex Chebyshev polynomial fraction. Through the partial-fraction expansion the complex Chebyshev polynomial fraction is decomposed into two sets of basic discrete-element models. The parameters in the models are obtained through the least-square curve-fitting to the available analytical or on-site measurement results. The accuracy and validity of the L-P model is validated through the applications to the surface circular foundations, embedded square foundations and pile group foundations, respectively. It is shown that in general the present model has better accuracy and needs fewer parameters than the existing L-P models.

1. Introduction

In the past few decades, the dynamic behavior of soil–structure interaction (SSI) has been studied by some researchers. The modeling to structures which exhibit the linear or nonlinear behavior has been well developed by the numerical methods such as the finite-element method. In contrast, the major challenge in SSI analysis is the modeling to dynamic behavior of the foundation sitting on soil. In some cases, the rigorous solutions of dynamic behavior of soil can be obtained from the three-dimensional analysis such as using the boundary-element method [1–2] or the scaled boundary finite-element method [3] and so on. However, in the practical applications, simple physical models in frequency domain (typically, the ratio of the amplitude of the applied load to the resulting displacement, which is called as impedance function) can be used. A few researchers have made tremendous effort to study the impedance functions for various types of foundations and soils [4–17].

For the simplification in application, the lumped-parameter (L-P) model with frequency-independence is commonly adopted to represent the impedance functions in the frequency domain. The L-P model is commonly composed of masses, springs, and dashpots. The parameters in the model are obtained by using an optimization technique to minimize the discrepancy between the impedance function from the approximate model and that from the rigorous theory or the on-site measurements. The advantage of the L-P model with frequency-independence is that it can be directly applied to the linear/nonlinear dynamic analysis of structures in the time domain. The L-P model with frequency-independence has been studied by Wolf and Somaini [18], De Barros and Luco [19], Takemiya [20], Luan and Lin [21]. Attempting to diminish the difficulty in expansion, Wolf [22–24] systematically developed a set of L-P models. Wu and his co-workers [25–27] developed another set of L-P models different from Wolf's models. Moreover, Safak [28] used the z-transform to obtain a representation of the impedance functions as a ratio of two simple polynomials.

Since Chebyshev first introduced the polynomial which bear his name in 1854 [29], the Chebyshev polynomial has taken a significant position in numerical analysis such as orthogonal polynomials, function approximation, numerical integration, and spectral methods for partial differential equations. The Chebyshev polynomials can be used to reduce the problem of wiggling of the approximation if using the simple polynomials of high degree and reduce the degree of the approximating polynomials with a minimal loss of accuracy [30]. This paper uses the complex Chebyshev polynomial fraction to describe the L-P model of foundations. The normalized flexibility function for foundations is studied in detail. Through the partial-fraction expansion, the complex Chebyshev polynomial fraction is decomposed into two

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sets of basic discrete-element models. The parameters in the models are obtained through the least-square curve-fitting to the available analytical or test results. The accuracy and validity of the L-P model is validated through the applications to the surface circular foundation, embedded square foundation and pile group foundation, respectively.

2. Definition of complex Chebyshev polynomial

The Chebyshev polynomial $T_n(x)$ is a polynomial in x of degree n (here, only the case when $n \geq 0$ is considered), defined by the relation [29]

$$T_n(x) = \cos n\theta, \quad \text{when} \quad x = \cos\theta \tag{1}$$

where x is real number. It is well known that $\cos n\theta$ is a polynomial of degree n in $\cos\theta$. We can easily deduce the fundamental recurrence relation from Eq. (1), by utilizing the trigonometric identity

$$\cos n\theta + \cos(n-2)\theta = 2\cos\theta\cos(n-1)\theta \tag{2}$$

as follows;

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \quad (n = 2, 3, \cdots) \tag{3a}$$

together with the initial conditions:

$$T_0(x) = 1, \quad T_1(x) = x \tag{3b}$$

Similarly, the complex Chebyshev polynomials are defined by the relation

$$\tilde{T}_0(z) = 1, \quad \tilde{T}_1(z) = z \tag{4a}$$

$$\tilde{T}_n(z) = (-1)^{n-1} 2z\tilde{T}_{n-1}(z) - \tilde{T}_{n-2}(z) \quad (n = 2, 3, \cdots) \tag{4b}$$

where $z = xi$ and $i = \sqrt{1}$. Using Eq. (4), polynomials $\{\tilde{T}_n(z)\}$ of any degree can be recursively generated. Table 1 gives the first ninth complex Chebyshev polynomials in terms of powers of z. The first eight complex Chebyshev polynomials are plotted in Fig. 1.

3. Dynamic flexibility functions of foundations

Consider a rigid foundation supported on soil and vibrated by a harmonic force. For harmonic excitation of frequency ω, the dynamic stiffness functions (the ratio of the amplitude of the applied load $P(\omega)$ to the resulting displacement $u(\omega)$) is expressed as

$$S(a_0) = K_s[k(a_0) + ia_0 c(a_0)] \tag{5}$$

where K_s represents the static stiffness of the foundation; $k(a_0)$ and $c(a_0)$ represent the dimensionless dynamic stiffness and dimensionless damping coefficients, respectively. In Eq. (5), a_0 is the dimensionless frequency parameter.

$$a_0 = \frac{\omega d}{V_s} \tag{6}$$

in which V_s is the shear wave velocity of soil and d is the characteristic length of the foundation (e.g., the radius of a circular foundation or the half-side length of a square foundation).

Base on the study of Wu and Lee [26], using the normalized flexibility function (the ratio of the resulting displacement $u(\omega)$ to the amplitude of the applied load $P(\omega)$) to describe the dynamic property of the foundation has some advantages: the problem of selecting the appropriate weights for different frequency ranges can be avoided because the dynamic flexibility function of foundation usually decreases with the frequency parameter so that the low-to-medium frequency equations are automatically emphasized without introducing any weights.

Without considering the couple effect among foundation stiffnesses in different directions, the dynamic flexibility function is just the reciprocal of the dynamic stiffness function. Therefore, the dynamic flexibility function $F(a_0)$ can be simply expressed as

$$F(a_0) = \frac{1}{S(a_0)} = F_s\bar{F}(a_0) \tag{7}$$

where $F_s = 1/K_s$ is the static flexibility and $\bar{F}(a_0) = 1/[k(a_0) + ia_0 c(a_0)]$ is the normalized flexibility function.

Table 1
The first ninth complex Chebyshev polynomials $\tilde{T}_n(z)$ (n=0, 1, 2,..., 8).

$\tilde{T}_0(z) = 1 = T_0(x)$
$\tilde{T}_1(z) = z = T_1(x)i$
$\tilde{T}_2(z) = -2z^2 - 1 = T_2(x)$
$\tilde{T}_3(z) = -4z^3 - 3z = T_3(x)i$
$\tilde{T}_4(z) = 8z^4 + 8z^2 + 1 = T_4(x)$
$\tilde{T}_5(z) = 16z^5 + 20z^3 + 5z = T_5(x)i$
$\tilde{T}_6(z) = -32z^6 - 48z^4 - 18z^2 - 1 = T_6(x)$
$\tilde{T}_7(z) = -64z^7 - 112z^5 - 56z^3 - 7z = T_7(x)i$
$\tilde{T}_8(z) = 128z^8 + 256z^6 + 160z^4 + 32z^2 + 1 = T_8(x)$

4. Flexibility functions approximated by complex Chebyshev polynomial fraction

In the present study, the normalized flexibility function of the foundation is approximately described by a ratio of two complex Chebyshev polynomials

$$\bar{F}(a_0) \approx \hat{F}(t) = \frac{\eta(t)}{\mu(t)} = \left[\frac{1 + \eta_1\tilde{T}_1(t) + \eta_2\tilde{T}_2(t) + \cdots \eta_N\tilde{T}_N(t)}{1 + D + \mu_1\tilde{T}_1(t) + \mu_2\tilde{T}_2(t) + \cdots \mu_N\tilde{T}_N(t) + G\eta_N\tilde{T}_{N+1}(t)}\right] \tag{8}$$

where $t = a_0 i / a_{0max}$, a_{0max} is the maximum value of the approximate range of frequency. η_N and μ_N are the coefficients of the numerator complex Chebyshev polynomials and the denominator complex Chebyshev polynomials, respectively. The coefficients D and G in the denominator polynomials should be deliberately selected such that the doubly asymptotic feature of $\bar{F}(a_0)$ can be ensured, i.e.

- It should be exact at the static limit, namely $\bar{F}(a_0) \to 1$ when $a_0 \to 0$.
- It should be exact at the high-frequency limit, namely $\bar{F}(a_0) \to 1/(ia_0\delta)$ when $a_0 \to a_{0\max}$.

Therefore, D and G should be respectively

$$D = \begin{cases} \sum_{l=1}^{N/2} (-1)^l \eta_{2l} - \sum_{l=1}^{N/2} (-1)^l \mu_{2l} & N = 2, 4, 6 \cdots \\ \sum_{l=1}^{(N-1)/2} (-1)^l \eta_{2l} - \sum_{l=1}^{(N-1)/2} (-1)^{l+1} \mu_{2l} + (-1)^{\frac{N-1}{2}} \delta a_{0max} \eta_N / 2 & N = 1, 3, 5 \cdots \end{cases} \tag{9}$$

$$G = \frac{\delta a_{0max}}{2} \tag{10}$$

where δ denotes the dimensionless damping coefficient at the high-frequency limit a_{0max}.