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Segmental multi-point linearization for parameter sensitivity approximation in reliability analysis

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ABSTRACT

This paper proposes an efficient method named segmental multi-point linearization for accurate approximation of the sensitivity of failure probability with respect to design parameters. The method is established based on a piecewise linear fitting of the limit state surface and the analytical integral of the gradient of the failure probability with respect to parameters in the limit state function. The proposed method presents an attractive ratio of accuracy to computational cost. The general framework is scalable such that, by adjusting its complexity, different levels of accuracy can be achieved. The method is especially suitable to be implemented in gradient-based routines for solving reliability-based design optimization problems where accuracy of the parameter sensitivity is essential for convergence to an optimal solution.

1. Introduction

The primary concern of reliability analysis is the approximation of the failure probability P_f or the equivalent generalized reliability index β, which is defined as $\beta = \Phi^{-1}(1 - P_f)$ [1], where $\Phi^{-1}(\cdot)$ is the inverse function of the cumulative distribution function (CDF) of the standard normal distribution. In addition, computation of sensitivity of failure probability to changes in parameters that influence the failure probability is an important aspect as this quantification provides insights on how the failure probability might change with changes in the values of some parameters. The partial derivatives of failure probability with respect to these parameters are called parameter sensitivity [1]. The parameters are categorized into two groups [2]. One group contains the parameters that appear in the probability density functions (PDF) of random variables and they are usually called distribution parameters. The other set of parameters, called design parameters, appear in the limit state function, a mathematical description that defines whether the component or system is safe or not. *In this paper, we propose an efficient numerical method to approximate the parameter sensitivity of the failure probability with respect to design parameters.*

The motivation of this work is due to the increasing demand for improvement of engineering designs based on reliability analysis, for example, either reducing cost without sacrificing reliability or simply increasing reliability of current design, which can be expressed mathematically as an optimization problem [11], called reliability-based design optimization (RBDO) [12–20]. In the RBDO formulation, the aforementioned design parameters are treated as design variables. Most RBDO problems are formulated as optimization problems with probabilistic constraint(s). The sensitivity of the probabilistic constraint is particularly important if one wants to use efficient gradient-based routines to solve the design optimization problem.

There are mainly two major gradient-based approaches for RBDO in the literature: the Reliability Index Approach (RIA), which explicitly uses the gradient of probabilistic constraint(s) in the optimization; and the Performance Measure Approach (PMA), which constructs target performance function(s) as equivalent deterministic constraint(s) by inverse reliability analysis [12], and thus the gradient of probabilistic constraint(s) is involved implicitly. Among different alternatives, currently the two approaches are mostly implemented in conjunction with first-order reliability method (FORM) and FORM-based expressions for sensitivity [13–19]. To improve the solutions, heuristic updates of the failure probability using, for example, second-order reliability method (SORM), Monte Carlo simulation (MCS), and other reliability methods, are used in many RBDO approaches [19,16], however, in general, the approximations of the sensitivity of failure probability in these approaches are still based on FORM. Another type of gradient-based approach employs the sample average

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approximation (SAA) where the failure probability and its sensitivity calculation are both carried out using MCS-based techniques [6], which require very high computational cost but could yield quite accurate estimation of the sensitivity.

Although numerical estimation of failure probability has been studied extensively, there are not many papers in the literature that focus on the numerical approximation of the parameter sensitivity. Early work can be found around the year 1990, mostly from a theoretical point of view. For instance, Hohenbinchler and Rackwitz [2] derived the expressions for parameter sensitivity of the estimated failure probability obtained by FORM with respect to both distribution and design parameters. In other words, the expressions are only exact when the limit state function is linear. Breitung [3] derived an analytical expression for the parameter sensitivity of failure probability with general limit state functions, and he suggested an asymptotic approximation, which has similar terms as the expression in [2]. There is also a study on the derivatives of probability functions done by Uryasev [4]. However, although analytical expressions for the parameter sensitivity of the failure probability are available, usually precise numerical evaluation of the sensitivity is not possible, similarly to the case of failure probability evaluation.

Some numerical methods have been developed to approximate the parameter sensitivity. An efficient method is available by implementing the expressions in [2], which are by nature an approximation. The advantage is that it is so simple that it can be computed with very low computational cost. However, the expressions can be quite inaccurate when the nonlinearity of the limit state function is significant. Karamchandani and Cornell [5] developed a method that approximates the parameter sensitivity with respect to distribution parameters that can take second order effect into account, based on SORM and the finite difference method. In addition, MCS-based stochastic methods are often used for approximating the sensitivity of probability [6–10]. For example, the sample average approximation was used for both component and system reliability constrained problems by Royset and Polak [6,8,9]. Recent theoretical advances by van Ackooij and Henrion [10] provided representations of the gradients to convex probability functions as integrals with respect to uniform distribution over the unit sphere.

In this paper, we focus on the approximation of the parameter sensitivity of component failure probability since this is the foundation for extensions to system reliability analysis. The proposed method of segmental multi-point linearization (SML) is established to estimate the sensitivity of failure probability with respect to design parameters. The method can be directly employed in the framework of RIA for RBDO. Thus the main purpose of this work is to provide an alternative method, that is more accurate than FORM-based approximation, but requires significantly less computational cost than approximations based on MCS, as illustrated by Fig. 1. Furthermore, the theory for the method allows many variations to match different requirements on accuracy for a variety of problems. This paper is tailored towards the implementation of the

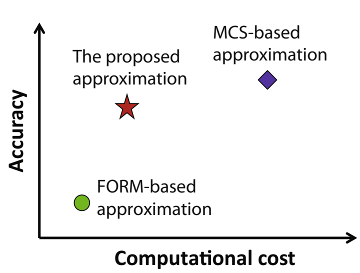

Fig. 1. Conceptual comparison of different approximation approaches for sensitivity of failure probability.

proposed method in the context of RBDO. *However, we would like to mention that the method for sensitivity analysis is independent from this particular application.* Three appendices supplement this paper. The nomenclature used is given in Appendix A. A brief derivation of the parameter sensitivity of the failure probability is given in Appendix B. A pseudo code for the orthogonal fitting SML is provided in Appendix C.

2. Importance of accurate estimation of sensitivity in RBDO

One problem with FORM-based approaches in RBDO is that the error in the approximate sensitivity of the probabilistic constraint can lead the optimization to converge to non-optimal solutions. This will be investigated by analyzing two popular approaches of RBDO, namely FORM-based RIA and PMA, via the well-known Karush–Kuhn–Tucker (KKT) optimality conditions of the approaches. Consider a generic formulation of RBDO problems with one reliability component:

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) \\ s.t. \quad & P_f = \int_{g(\mathbf{v},\mathbf{x})<0} f_{\mathbf{v}}(\mathbf{v})\mathrm{d}\mathbf{v} \leqslant P_f^t \\ & \mathbf{h}(\mathbf{x}) \leqslant \mathbf{0} \end{aligned} \tag{1}$$

where P_f^t is the target failure probability; $\mathbf{x}$ is the vector of design variables (i.e. design parameters); $\mathbf{v}$ is the vector of random variables with distribution described by the PDF $f_{\mathbf{v}}(\mathbf{v})$; $f(\mathbf{x})$ is the objective function; $g(\mathbf{v},\mathbf{x})$ is the limit state function; and $\mathbf{h}(\mathbf{x})$ is a set of deterministic constraints such as lower and upper bounds of $\mathbf{x}$. Reliability methods often require a probability preserving transformation $\mathbf{u} = T(\mathbf{v})$, where $\mathbf{u}$ is a vector of independent standard normal random variables. Replacing $\mathbf{v}$ by $T^{-1}(\mathbf{u})$ defines the limit state function in an equivalent form $G(\mathbf{u},\mathbf{x}) = g(T^{-1}(\mathbf{u}),\mathbf{x})$, which is the function that is eventually used for reliability analysis. The formulation (1) would accordingly become:

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) \\ s.t. \quad & P_f = \int_{G(\mathbf{u},\mathbf{x})<0} \varphi_n(\mathbf{u})\mathrm{d}\mathbf{u} \leqslant P_f^t \\ & \mathbf{h}(\mathbf{x}) \leqslant \mathbf{0} \end{aligned} \tag{2}$$

where $\varphi_n(\cdot)$ is the multi-variate standard normal PDF with uncorrelated elements and n is the number of random variables. Equivalently, the constraint on failure probability can be expressed in terms of generalized reliability index, which is defined as $\beta = \Phi^{-1}(1 - P_f)$.

Mathematically, the KKT optimality conditions of the RBDO model described in (2), are:

(1) Stationarity condition: $\nabla_{\mathbf{x}} f + \lambda \nabla_{\mathbf{x}} P_f + \sum \gamma_i \nabla_{\mathbf{x}} h_i = 0$
(2) Primal feasibility: $P_f - P_f^t \leqslant 0, h_i \leqslant 0\ \forall i$
(3) Dual feasibility: $\lambda \geqslant 0, \gamma_i \geqslant 0\ \forall i$
(4) Complementary slackness: $\lambda(P_f - P_f^t) = 0, \gamma_i h_i = 0\ \forall i$

where λ and γ_i's are the Lagrange multipliers. The KKT conditions are necessary for a solution to be optimal. Because for most RBDO cases, P_f and $\nabla_{\mathbf{x}} P_f$ can only be evaluated approximately, then the KKT conditions are only approximately satisfied at the optimized solution. Different RBDO algorithms have different approximations about the KKT conditions in the end. Thus they may converge to different solutions. However, when the probabilistic constraint is active in an optimization problem, PMA and RIA will converge to the same solution if they are both based on FORM as they share the same approximations of the KKT conditions.