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Strategies for finding the design point under bounded random variables

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A B S T R A C T

The First Order Reliability Method is well accepted as an efficient way to solve structural reliability problems with linear or moderately non-linear limit state functions. Bounded random variables introduce strong non-linearities in the mapping to standard Gaussian space, making search for design points more demanding. Since standard Gaussian space is unbounded, two particular problems have to be addressed: a. the limiting behavior of the probability mapping as a random variable approaches its upper or lower limits; b. how to impose the bounds if design point search leaves the problems supporting domain. Both problems have been overlooked elsewhere, and are addressed in the present article. Based on the Principle of Normal Tail Approximation, two alternatives for the mapping to standard Gaussian space are studied. Three different schemes are proposed to impose the limits of bounded random variables, in the reverse mapping to original design space. Several algorithms are investigated with respect to their ability to find the design point in highly non-linear problems involving bounded random variables. A challenging benchmark reliability problem is also presented herein, and used as a test bed to explore the proposed strategies and the performance of the optimization algorithms.

1. Introduction

The First Order Reliability Method is well accepted as an efficient way to solve structural reliability problems with linear or moderately non-linear limit state functions. High non-linearity is introduced in reliability problems by non-linear mechanical responses [1], but also by correlation between the random variables and by highly non-Gaussian probability distributions. Correlation, non-Gaussianity and generally bounded distributions introduce non-linearities in the mapping to standard Gaussian space, making the search for the design point more demanding.

Several algorithms for structural reliability analysis using FORM have been proposed [2–12]. This includes simple, mathematical programming algorithms [2–8] as well as heuristic algorithms such as particle swarm optimization [9] or genetic algorithms [10,11]. Techniques for solving structural reliability problems using FORM also include the use of response surfaces [12–19]. Many of these algorithms and techniques have found their way into well-established structural reliability software [20–22]. However, none of the cited articles [2–22] addresses the specific challenges in finding the design point in problems involving uniform or other bounded random variables. Instability and convergence difficulties were identified by Youn and Choi [23], in reliability-based design optimization problems involving uniform and non-Gaussian distributions. For problems involving only (multi-dimensional) uniform random variables, optimization techniques that look for the most probable failure point in original design space [24] cannot be used, as all points are equally likely. Hence, the specific difficulties of finding the design point in problems involving uniform or generally bounded random variables have not been consistently addressed in the published literature.

This article discusses challenges in finding the design point in problems involving uniform and other bounded random variables. The discussion covers the Principle of Normal Tail Approximation [25,26], and the mapping to the standard Gaussian space. Two alternatives for this mapping are considered: a direct and theoretically reversible mapping, and a formal but potentially irreversible mapping. These mappings are studied as the interactive design point search moves outside of the support domain (some random variable is pushed beyond its lower or upper limit).

As standard Gaussian space is unbounded, during design point search any bounded random variable could potentially exceed its limits, causing breakdown of the computations. Hence, bounds have to be imposed when mapping random variables back to the original design space. Three different schemes are presented in this article to impose the limits of bounded random variables: bisection,

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reflection and limit. The three schemes are tested in combination with the direct and the formal probability mapping.

A challenging academic problem is also presented in the article. It involves a simply-supported beam subject to a concentrated load of random intensity. The load can occupy a random position over the beam, following an uniform distribution. Although the underlying mechanical problem is very simple, the uniform distribution introduces strong non-linearities, which makes finding the design point a very challenging task. An N-dimensional version of the same problem, involving $N/2$ concentrated loads, is also considered.

It is shown that the Hasofer–Lind–Rackwitz–Fiessler (HLRF) algorithm fails to converge for most configurations of the proposed problems. Hence, the improved HRFL (iHLRF) and Sequential Quadratic Programming (SQP) algorithms are also employed in the solutions. The performance of these algorithms is investigated with respect to two mappings to standard Gaussian space and to the three schemes to impose the bounds in the reverse probability mapping. In total, twenty-eight configurations of the proposed benchmark problem are investigated. The article is finished with relevant conclusions with respect to the several alternatives proposed herein for the solution of reliability problems involving bounded random variables.

2. FORM and probability mapping

2.1. The First Order Reliability Method (FORM)

Let $\mathbf{X}=[X_1,X_2,\ldots,X_n]^t$ be a random variable vector describing the uncertainties in geometry, material properties and loading, where n is the number of random variables and $[\cdot]^t$ is the transpose operator. A limit state function $g_{\mathbf{x}}:\mathbb{R}^n\rightarrow\mathbb{R}$ is written in such a way as to divide the random variable domain in safety and failure domains:

$$\begin{aligned}\Omega_f&=\{\mathbf{x}|g_{\mathbf{x}}(\mathbf{x})\leqslant 0\}\ \text{is the failure domain;}\\ \Omega_s&=\{\mathbf{x}|g_{\mathbf{x}}(\mathbf{x})>0\}\ \text{is the safety domain.}\end{aligned}\tag{1}$$

The failure probability is given by:

$$P_f=P[\mathbf{X}\in\Omega_f]=\int_{\Omega_f}f_{\mathbf{X}}(\mathbf{x})d\mathbf{x}\tag{2}$$

where $f_{\mathbf{X}}(\cdot)$ denotes the joint probability density of random vector $\mathbf{X}$. Also, assume that some variables in $\mathbf{X}$ have limited support on the real line (i.e., they are bounded), such that: $\{x_a\leqslant x\leqslant x_b\}$.

The First Order Reliability Method (FORM) is generally accepted as an efficient way to solving Eq. (2), for low dimensions of vector $\mathbf{X}$ and for linear or moderately non-linear limit state functions $g_{\mathbf{x}}(\cdot)$. In the FORM method, Eq. (2) is solved indirectly, by introducing a convenient mapping from the space of the original random vector $\mathbf{X}$ (x-space) to the so-called standard Gaussian space (u-space) [25,27–29]:

$$\mathbf{u}=\mathbb{T}(\mathbf{x}),g(\mathbf{u},t)=g_{\mathbf{x}}(\mathbb{T}^{-1}(\mathbf{u}),t),\tag{3}$$

where all components of vector $\mathbf{u}$ are independent and identically distributed standard Gaussian random variables. This mapping can be accomplished by means of the Principle of Normal Tail Approximation [25] and by the Nataf transformation [27–29], to be described.

In standard space, the joint probability density $f_{\mathbf{U}}(\cdot)$ is rotationally symmetric: hence, the point $\mathbf{u}^*$ on the limit state function $g(\cdot)$ which is closest to the origin represents the most probable failure point, also known as **design point**. This feature allows search for the design point to be cast as a constrained optimization problem:

$$\begin{cases}\mathbf{u}^*=\arg\ \min\{\frac{1}{2}\mathbf{u}^t\mathbf{u}\};\\ \text{subject to }g(\mathbf{u})=0.\end{cases}\tag{4}$$

From Eq. (4), $\beta=\|\mathbf{u}^*\|=(\mathbf{u}^{*t}\mathbf{u}^*)^{1/2}$ is the so-called Hasofer–Lind reliability index [30], which comes to be the distance between $\mathbf{u}^*$ and the origin of standard Gaussian space. The FORM method, therefore, consists in finding the design point $\mathbf{u}^*$ and approximating the original limit state function $g(\mathbf{u})$ by a tangent hypersurface at the design point. Hence, the first-order approximation of the failure probability becomes:

$$P_f=P[g(\mathbf{U})\leqslant 0]\simeq\Phi(-\beta);\tag{5}$$

where $\Phi(\cdot)$ is the standard Gaussian cumulative probability distribution function.

For the highly non-linear problems to be considered herein, the first order approximation of the failure probability is unlikely to be accurate. However, it is still important to be able to find the design point, for instance, in order to perform Monte Carlo simulation with importance sampling using the design point [31] or to construct precise response surface approximations centered at the design point [14,15,19].

It is important to point out that the concept of a design point as the most probable failure point is theoretical, and associated to the transformation to standard Gaussian space. For instance, for (multi-dimensional) problems involving only uniform random variables, all points in the failure domain are equally likely in x-space... However, the concept and the usual interpretations remain valid in u-space.

2.2. Normal tail approximation

The Principle of Normal Tail Approximation [25] allows a non-Gaussian variable X to be transformed in an "equivalent" Gaussian variable Y by means of a point-wise probability mapping:

$$F_Y(x^*)=F_X(x^*).\tag{6}$$

The mapping in Eq. (6) is applied, in scalar fashion, to the marginal distributions. Eq. (6) is a probability-preserving equation, which makes the probability content to the "left" and "right" of point x^* to remain unaltered in the mapping. However, two parameters (μ_Y and σ_Y – mean and standard deviation) of the "equivalent" Gaussian variable Y have to be determined. Hence, an additional equation is required. A second equation, used herein to complete the mapping, is [25]:

$$f_Y(x^*)=f_X(x^*).\tag{7}$$

This equation is completely arbitrary, and is not required for probability preservation. Other equations and rules for the probability mapping have also been presented elsewhere [26].

Introducing the Hasofer–Lind transformation [30]:

$$u^*=\frac{x^*-\mu_Y}{\sigma_Y},\tag{8}$$

the mapping can be accomplished directly to standard Gaussian space. Manipulating Eqs. (6)–(8), well-known equations are obtained for the desired parameters (μ_Y and σ_Y):

$$u^*=\Phi^{-1}(F_X(x^*));\tag{9}$$

$$\sigma_Y=\frac{\phi(u^*)}{f_X(x^*)};\tag{10}$$

$$\mu_Y=x^*-u^*\sigma_Y.\tag{11}$$