Structural Safety 58 (2016) 94–104

Contents lists available at ScienceDirect

Structural Safety

journal homepage: www.elsevier.com/locate/strusafe

Global response sensitivity analysis of uncertain structures

G. Greegar, C.S. Manohar *

Department of Civil Engineering, Indian Institute of Science, Bangalore 560 012, India

ARTICLE INFO

Article history:
Received 22 November 2014
Received in revised form 26 September 2015
Accepted 26 September 2015

Keywords:
Global sensitivity analysis
Probability distance measures
Hausdorff distance
Intervals
Convex function models
Fuzzy variables

ABSTRACT

The problem of characterizing global sensitivity indices of structural response when system uncertainties are represented using probabilistic and (or) non-probabilistic modeling frameworks (which include intervals, convex functions, and fuzzy variables) is considered. These indices are characterized in terms of distance measures between a fiducial model in which uncertainties in all the pertinent variables are taken into account and a family of hypothetical models in which uncertainty in one or more selected variables are suppressed. The distance measures considered include various probability distance measures (Hellinger, l_2, and the Kantorovich metrics, and the Kullback–Leibler divergence) and Hausdorff distance measure as applied to intervals and fuzzy variables. Illustrations include studies on an uncertainly parametered building frame carrying uncertain loads.

1. Introduction

Consider an engineering mechanics problem in which a scalar output variable Y is related to the $n \times 1$ input vector $\boldsymbol{X}$ through the model $Y = f(\boldsymbol{X})$. The problem on hand could involve linear/nonlinear mechanical model and the behavior could be either static or dynamic. It is assumed that the vector $\boldsymbol{X}$ collectively represents all the sources of uncertainties in the problem. Given that the input $\boldsymbol{X}$ is uncertain in nature, the output Y would also be uncertain in nature. The problem of global sensitivity analysis, when $\boldsymbol{X}$ is modeled as a vector of random variables, seeks to answer questions of the following type:

(a) Can the contributions made by various components of $\boldsymbol{X}$ to the variance of Y be delineated? In doing so, can we take into account dependencies among components of $\boldsymbol{X}$? Can the contributions from groups of input random variables be delineated instead of segregating contributions from individual random variables?
(b) Can questions similar to the above can be tackled, if instead of variance of Y, we focus attention on the probability density function (pdf) $p_Y(y)$ or area under the pdf $p_Y(y)$ over specified segments (like, for example, segments of the tail region)?

The Sobol analysis provides answer to the first of these questions under the assumption that the vector $\boldsymbol{X}$ is made up of independent and identically distributed (iid) random variables distributed uniformly over 0–1 [1–6]. The analysis here leads to the decomposition of the variance of Y into contributions from each of the components of $\boldsymbol{X}$ acting as singlets, interacting pairs, and higher order interactions among $3, 4, \ldots, n$ variables. This is achieved by expanding $f(\boldsymbol{X})$ into a finite series of $2^n - 1$ terms involving a set of orthogonal functions of components of $\boldsymbol{X}$. This set of functions itself is deduced from the given $f(\boldsymbol{X})$. The method allows analysis with respect to groups of input variables. The analysis can be extended in a straightforward manner to the case when $\boldsymbol{X}$ consists of independent but not identical and non-uniformly distributed random variables [7]. When elements of $\boldsymbol{X}$ are dependent, the decomposition of $f(\boldsymbol{X})$ into a finite series of orthogonal functions is no longer feasible and the Sobol's analysis, in this sense, becomes inapplicable. The generalizations in such situations to consider correlation/dependence among elements of $\boldsymbol{X}$ in estimating the sensitivity indices have been discussed by Li et al. [8], Kucherenko et al. [9], Cui et al. [10], and Zhang et al. [11]. The study by Cao et al. [12] considers the application of Sobol's analysis to problems in which the uncertain inputs are modeled as random processes. The studies by Arwade et al. [7], Blatman and Sudret [13], and Mukherjee et al. [14] contain applications of Sobol's analysis to structural engineering problems. A review of computational aspects related to the evaluation of global sensitivity indices has been presented by Sudret [15]. The works of Li et al. [16], Rahman [17], Yadav and Rahman [18,19] contain extensive accounts of the

* Corresponding author. Tel.: +91 80 2293 3121.
E-mail addresses: greegarg@civil.iisc.ernet.in (G. Greegar), manohar@civil.iisc.ernet.in (C.S. Manohar).

http://dx.doi.org/10.1016/j.strusafe.2015.09.006
0167-4730/

application of meta-modeling techniques in modeling global response sensitivity indices.

The second class of problems has been investigated by Liu et al. [20] who employ a directed distance measure, namely, the Kullback–Leibler (KL) entropy, to measure the distance between two probabilistic models. Here the pdf of the specified response variable is evaluated under two alternative scenarios, namely, (a) when uncertainty in all variables are included, and (b) when the uncertainty in a chosen variable is suppressed. These two scenarios lead to two different models for pdf of the response and the distance between these models would provide an idea of impact of uncertainty in the chosen input parameter on the response variable of interest. These authors also detail how the proposed procedure can be extended to assess sensitivity of probability of response variable lying in a prescribed interval. One of the notable features of this approach is that the analysis permits treatment of input variables being non-Gaussian and dependent.

In a recent study, the present authors have explored the application of alternative probability distance measures, including the l_2 norm and the Hellinger metric, in developing definitions for global sensitivity indices [21]. This study has shown that, for the case when $\boldsymbol{X}$ is made up of an iid sequence of random variables, the method based on the l_2 norm is related to the Sobol analysis and is able to exactly reproduce all the Sobol indices. The method remains valid for more general models for $\boldsymbol{X}$ involving non-Gaussian and dependent random variables. Thus, the l_2 norm based analysis, with altogether different moorings than the Sobol method, is perceived as a generalization of the Sobol analysis. Furthermore, the authors have also explored the use of the Hellinger distance, which is a metric (unlike the KL divergence), in pdf-based global sensitivity analysis. Extension of these ideas to problems of global response sensitivity analyses when inputs are modeled as a set of random processes has been explored by Abhinav and Manohar [22].

It is observed that problems of global sensitivity analysis seem to have been tackled in the existing literature only within the framework of probabilistic methods for uncertainty modeling. In situations wherein adequate data on uncertain parameters are unavailable, alternative frameworks for uncertainty modeling (including interval analysis, convex function modeling and fuzzy variables) become relevant (see, for example, [23–30]). Questions on how to perform global response sensitivity analysis in such situations remains to be addressed. In this context, the present study notes that, akin to the notion of various probability distance measures, as defined for random variables, there exist notions of model distance in the context of intervals and fuzzy sets. These distance measures can form the basis on which global sensitivity measures can be developed within non-probabilistic modeling framework. Further questions on treatment of mixed models of uncertainty, in which, both probabilistic and non-probabilistic methods are used for uncertainty modeling within a single problem, also merit attention. Also, within the context of probabilistic uncertainty modeling, we note that the definitions of the KL entropy and Hellinger distance measures involve the pdf of response. There exist other measures, such as the Kantorovich measure, which are defined in terms of the cumulative distribution function (cdf) and evaluation of global sensitivity indices based on these measures appears computationally easier. The exploration of these issues forms the subject matter of this paper. It may be noted that, the starting point in the present study has been the assumption that, in a given problem, non-probabilistic or combined non-probabilistic and probabilistic models for uncertainties have been deemed necessary. We do not specifically discuss contexts in which such approaches are considered appropriate. A discussion on these issues can be found in the works of Ben Haim and Elishakoff [23], and Moller and Beer [31].

2. Nature of response of uncertain structures

We consider the input–output relation of the form $Y = f(\boldsymbol{X})$ and take that the $n \times 1$ vector $\boldsymbol{X}$ is made up of components which are modeled either probabilistically or non-probabilistically. The non-probabilistic framework is taken to include intervals, convex functions, and (or) fuzzy variables. Thus, we represent $\boldsymbol{X}$ as $\boldsymbol{X} = (\Theta, \Psi, \Xi, \Gamma)^t$ where the superscript t denotes matrix transposition and the meaning of Θ, Ψ, Ξ, and Γ is summarized in Table 1. Given the uncertainty in $\boldsymbol{X}$, Y would also be uncertain in nature with the variability in Y being made up of contributions from different components of $\boldsymbol{X}$, their inter-dependencies, and mutual interactions as they get transformed through the input–output relation $Y = f(\boldsymbol{X})$. The objective of the present study is to quantitatively describe these contributions from each of the input variables with a view to characterize the relative influence of components of $\boldsymbol{X}$ on the variability in Y.

Remarks.

1. The model $Y = f(\boldsymbol{X})$ is fairly general in the sense that it allows for linear/nonlinear mechanical behavior and static/dynamic responses. For example,
   (a) Y could be one of the natural frequencies of a linear multi-degree of freedom (mdof) system with uncertain mass and stiffness properties.
   (b) Y could be the maximum principal stress at a critically stressed point in a nonlinear frame with uncertain geometry and (or) constitutivity and subject to uncertain loads.
   (c) Y could be the critical buckling load factor of a tower structure, or
   (d) Y could be the maximum displacement of a building frame with uncertain mass, stiffness, and damping properties and subjected to an earthquake load; the load here could be a random process and the vector $\boldsymbol{X}$ in this case would include random variables resulting from a discretization of the excitation process.
2. The mathematical nature of Y changes depending upon the nature of the input vector $\boldsymbol{X}$. For the special cases of $\boldsymbol{X} = \Theta$, $\boldsymbol{X} = \Psi$, $\boldsymbol{X} = \Xi$, and $\boldsymbol{X} = \Gamma$, it follows that the functions $f(\Theta), f(\Psi), f(\Xi)$, and $f(\Gamma)$, would respectively be a random variable, an interval,

Table 1
Description of uncertain variables.

No	Quantity	Size	Nomenclature	Specification
1	Θ	$n_\Theta \times 1$	Vector of random variables	n_Θth order joint probability density function $p_\Theta(\theta)$; $\Theta_i, i = 1, 2, \ldots, n_\Theta$ are, in general, dependent, and non-Gaussian
2	Ψ	$n_\Psi \times 1$	A set of intervals	$\underline{\Psi}_i \leqslant \Psi_i \leqslant \bar{\Psi}_i, i = 1, 2, \ldots, n_\Psi$
3	Ξ	$n_\Xi \times 1$	Variables modeled using a convex function.	$\Xi_i \in \Omega(\Xi) \forall\ i = 1, 2, \ldots, n_\Xi$ where $\Omega(\Xi_1, \Xi_2, \ldots, \Xi_{n_\Xi})$ is a convex function
4	Γ	$n_\Gamma \times 1$	Fuzzy variables	$\Gamma \in A = \{[\Gamma, \mu_A(\Gamma)],\ 0 \leqslant \mu_A(\Gamma) \leqslant 1\}\ \mu_A(\Gamma) =$ membership function

Note: $n_\Theta + n_\Psi + n_\Xi + n_\Gamma = n$.