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FORM, SORM, and spatial modeling in geotechnical engineering

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ABSTRACT

An intuitive ellipsoidal perspective is described together with three spreadsheet-automated constrained optimizational FORM procedures and a SORM approach. The three FORM procedures are then compared in the context of geotechnical examples of a confined soil element, a rock slope, and an embankment on soft ground with spatially autocorrelated undrained shear strength in the soft clay foundation, the performance function of which is based on a reformulated Spencer method with search for reliability-based critical noncircular slip surface. Two methods of modeling spatial autocorrelations are presented, and the merits and limitations of the three constrained optimizational FORM procedures are studied. The complementary roles and interconnections among the three constrained optimizational FORM procedures and SORM approach are emphasized. Comparisons are also made with Monte Carlo simulations.

1. Introduction

The Hasofer-Lind [13] index for cases with correlated normal random variables and the first-order reliability method (FORM) for cases with correlated nonnormals are well explained in Ditlevsen [10], Shinozuka [39], Ang and Tang [1], Melchers [34], Haldar and Mahadevan [12], and Baecher and Christian [2], for example. The potential inadequacies of the FORM in some cases have been recognized, and more refined alternatives proposed, in Chen and Lind [7], Der Kiureghian et al. [9], Wu and Wirsching [47], and Zhao and Ono [49], among others. On the other hand, the usefulness and accuracy of the FORM in most applications are well recognized, for instance by Rackwitz [38].

The focus of this paper is on spreadsheet-based procedures for FORM (which extends the Hasofer–Lind index for correlated normals into the nonnormal realm), SORM on the foundation of FORM results, system FORM, and reliability analysis accounting for spatially autocorrelated soil properties. Specifically, a simple geomechanics example is first examined to illustrate spreadsheet based SORM analysis on the foundation of FORM reliability index and FORM design point. This is followed by a rock slope with correlated nonnormal random variables, solved using the **u** space approach for comparison with the Low and Tang [29] **n** space approach. Finally, spatially autocorrelated shear strength is modeled in the reliability analysis of an embankment on soft ground. The advantages and limitations of three FORM computational approaches, namely constrained optimization with respect to the original random variables, the normalized but unrotated **n** vector, and the normalized and rotated **u** vector, respectively, are investigated.

Spatial autocorrelation (also termed spatial variability) arises in geological material by virtue of its formation by natural processes acting over unimaginably long time (millions of years). This endows geomaterial with some unique statistical features (e.g. spatial autocorrelation) not commonly found in structural material manufactured under strict quality control. For example, by the nature of the slow precipitation (over many seasons) of fine-grained soil particles under gravity in water in nearly horizontal layers, two points in close vertical proximity to one another are likely to be more positively correlated (likely to have similar undrained shear strength c_u values, for example) than two points further apart in the vertical direction.

System FORM and SORM are extensions of FORM. The classical computational approach of FORM in normalized and rotated **u**-space is elegant but shrouded in mathematical details. An intuitive perspective and two spreadsheet-automated FORM computational approaches were provided in Low and Tang [28,29], with the aim to facilitate understanding. The two approaches (in the **x** space and **n** space, respectively) are summarized in the next section, together with a third alternative of spreadsheet constrained optimization in the **u** space. They are meant to complement the elegant classical **u**-space FORM approach. The Low and Tang [29] **n**-space approach easily reverts to the **u**-space (required for SORM computation) using a simple equation.

The spreadsheet-based reliability approaches can be applied to implicit functions or stand-alone numerical packages, via the response surface methods (RSM), for example. RSM as a bridge between spreadsheet-based geotechnical FORM analysis and stand-alone numerical packages is illustrated by the following:

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(i) Xu and Low [48] conducted FORM analysis on second-degree polynomial response surface function constructed from finite element analysis of embankment stability.
(ii) Chan and Low [6] constructed second-degree polynomial response surface based on finite element analysis of a laterally loaded pile, and conducted FORM and SORM analyses.
(iii) Lü and Low [32] conducted FORM and SORM analyses on the movement of a horseshoe-shaped highway tunnel, in which the response surface function was based on numerical analyses using the code FLAC.

2. Intuitive perspective and efficient spreadsheet approaches for FORM and SORM

The matrix formulation [45,10] of the Hasofer and Lind [13] index β is:

$$\beta = \min_{\mathbf{x}\in F} \sqrt{(\mathbf{x}-\boldsymbol{\mu})^T \mathbf{C}^{-1} (\mathbf{x}-\boldsymbol{\mu})} \tag{1a}$$

$$\text{or, equivalently}: \beta = \min_{\mathbf{x}\in F} \sqrt{\left[\frac{x_i-\mu_i}{\sigma_i}\right]^T \mathbf{R}^{-1} \left[\frac{x_i-\mu_i}{\sigma_i}\right]} \tag{1b}$$

where $\boldsymbol{x}$ is a vector representing the set of random variables x_i, $\boldsymbol{\mu}$ the vector of mean values μ_i, $\boldsymbol{C}$ the covariance matrix, $\boldsymbol{R}$ the correlation matrix, σ_i the standard deviations, and F the failure domain. Low and Tang [27] used Eq. (1b) instead of Eq. (1a), because the correlation matrix $\boldsymbol{R}$ is easier to set up, and conveys the correlation structure more explicitly than the covariance matrix $\boldsymbol{C}$. The point denoted by the x_i values**,** which minimize Eq. (1) and satisfies $\mathbf{x} \in F$, is the design point**.** This is the point of tangency of an expanding dispersion ellipsoid with the LSS, which separates safe combinations of parametric values from unsafe combinations (Fig. 1). The one-standard-deviation ($1-\sigma$) dispersion ellipse and the β-ellipse in Fig. 1 are tilted by virtue of cohesion c and friction angle ϕ being negatively correlated. The quadratic form in Eq. (1) appears also in the negative exponent of the established probability density function of the multivariate normal distribution. As a multivariate normal dispersion ellipsoid expands from the mean-value point, its expanding surfaces are contours of decreasing probability values. Hence, to obtain β by Eq. (1) means maximizing the value of the multivariate normal probability density function, and is graphically equivalent to finding the smallest ellipsoid tangent to the LSS at the most probable failure point (the *design point*). This intuitive and visual understanding of the *design point* is consistent with the more mathematical approach in Shinozuka [39], in which all variables were standardized and the limit state equation was written in terms of standardized variables.

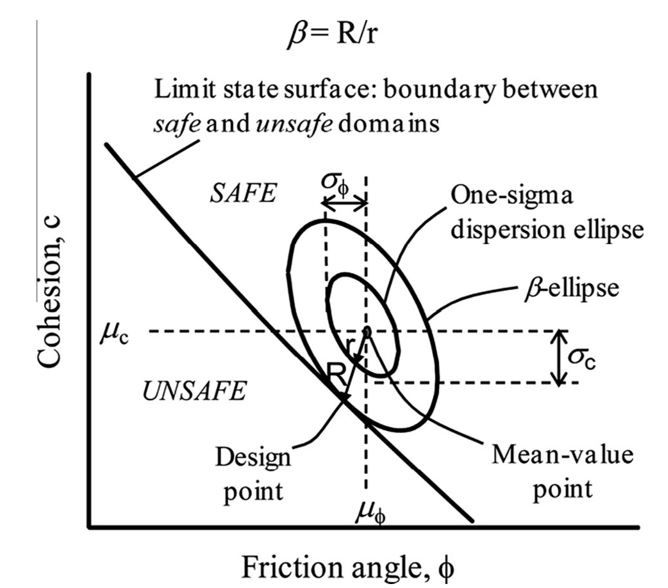

Fig. 1. Illustration of the reliability index β in the plane when c and ϕ are negatively correlated.

In FORM, one can rewrite Eq. (1b) as follows (Low and Tang [28]), and regard the computation of β as that of finding the smallest equivalent hyperellipsoid (centred at the equivalent normal mean-value point μ^N and with equivalent normal standard deviations σ^N) that is tangent to the limit state surface (LSS):

$$\beta = \min_{\mathbf{x}\in F} \sqrt{\left[\frac{x_i-\mu_i^N}{\sigma_i^N}\right]^T \mathbf{R}^{-1} \left[\frac{x_i-\mu_i^N}{\sigma_i^N}\right]} \tag{2}$$

where μ_i^N and σ_i^N can be calculated by the Rackwitz and Fiessler [37] transformation. Hence, for correlated nonnormals, the ellipsoid perspective still applies in the original coordinate system, except that the nonnormal distributions are replaced by an equivalent normal ellipsoid, centered not at the original mean values of the nonnormal distributions, but at the equivalent normal mean $\boldsymbol{\mu}^N$:

Eq. (2) and the Rackwitz–Fiessler equations were used in the spreadsheet-automated constrained optimization computational approach of FORM in Low and Tang [28]. An alternative to the 2004 FORM procedure is given in Low and Tang [29], which uses the following equation for the reliability index β:

$$\beta = \min_{\mathbf{x}\in F} \sqrt{\mathbf{n}^T \mathbf{R}^{-1} \mathbf{n}} \tag{3}$$

The computational approaches of Eqs. 1b, 2, and 3 and associated ellipsoidal perspective are complementary to the classical **u**-space computational approach, and may help overcome the conceptual and language barriers which Whitman [46] rightly noted.

The two spreadsheet-based computational approaches of FORM are compared in Fig. 2. Either method can be used as an alternative to the classical **u**-space FORM procedure. A third alternative is also shown in Fig. 2, for which the Microsoft Excel's built-in constrained optimization routine (*Solver*) is invoked to automatically vary the **u** vector so that β and the design point are obtained. This requires only adding one **u** column, and expressing the unrotated **n** vector in terms of **u**, where **u** is the uncorrelated standard equivalent normal vector in the rotated space of the classical mathematical approach of FORM. The vectors **n** and **u** can be obtained from one another, $\mathbf{n} = \mathbf{Lu}$ and $\mathbf{u} = \mathbf{L}^{-1}\mathbf{n}$, as follows (e.g., [31]:

$$\beta = \min_{\mathbf{x}\in F} \sqrt{\mathbf{n}^T \mathbf{R}^{-1} \mathbf{n}} = \min_{\mathbf{x}\in F} \sqrt{\mathbf{n}^T (\mathbf{LU})^{-1} \mathbf{n}} = \min_{\mathbf{x}\in F} \sqrt{(\mathbf{L}^{-1}\mathbf{n})^T (\mathbf{L}^{-1}\mathbf{n})} \tag{4a}$$

$$\text{i.e.}\quad \beta = \min_{\mathbf{x}\in F} \sqrt{\mathbf{u}^T \mathbf{u}}\quad \text{, where}\quad \mathbf{u} = \mathbf{L}^{-1}\mathbf{n} \quad\text{and}\quad \mathbf{n} = \mathbf{Lu} \tag{4b}$$

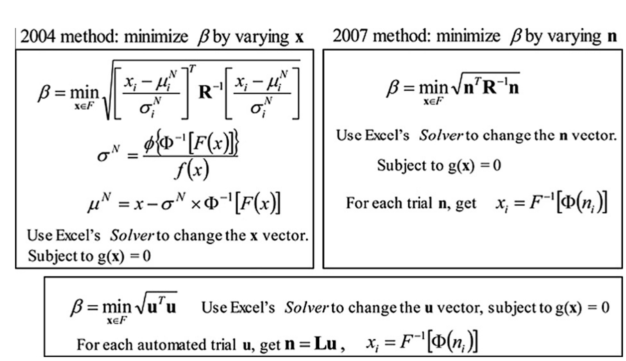

Fig. 2. Comparison of the two FORM computational approaches of Low and Tang (2004, 2007), and the additional **u**-to-**n**-to-**x** approach illustrated in this paper.