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Interval importance sampling method for finite element-based structural reliability assessment under parameter uncertainties

Hao Zhang *

School of Civil Engineering, University of Sydney, NSW 2006, Australia

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ABSTRACT

Parameters of a probabilistic model often cannot be determined precisely on the basis of limited data. In this case the unknown parameters can be introduced as intervals, and the imprecise probability can be modeled using a probability bounding approach. Common methods for bounding imprecise probability involve interval analysis to compute bounds of the limit state probability. A large number of interval finite element (FE) analyses have to be performed if the structural response defined as the limit state is determined implicitly through FE analysis. A new interval importance sampling method is developed in this paper which applies importance sampling technique to the imprecise probability. The proposed methodology has a desirable feature that expensive interval analyses are not required. Point samples are generated according to the importance sampling function. The limit states are computed using deterministic FE analyses. The bounds of the imprecise probability density function are introduced in the formulation at a later stage to incorporate the effects of the imprecision in the probability functions on the reliability results. Examples are given to illustrate the accuracy and efficiency of the interval importance sampling method. The second example also compares the proposed method with the conventional Bayesian approach.

1. Introduction

In structural reliability theory, the calculation of probability of failure (or, limit state probability), P_f, requires the evaluation of the multivariate integration:

$$P_f = \int_{g(X)\leqslant 0} f_X(X)dX = \int_{\mathbb{R}^s} \mathbf{I}[g(X) \leqslant 0] f_X(X)dX, \quad (1)$$

where $X = (X_1, \ldots, X_s)$ is the s-dimensional random vector representing uncertain quantities such as applied loads, structural strength and stiffness. $f_X(X)$ represents the joint probability density function for X. $g(X)$ is the limit state function and failure occurs when $g(X) \leqslant 0$. $\mathbf{I}[\ ]$ is the indicator function, having the value 1 if [ ] is "true" and the value 0 if [ ] is "false".

A key step in the evaluation of realistic limit state probability is to identify the proper distribution $f_X(X)$ for the basic random variables. When available data on structural strength and loads are limited, as is often the case in practice, statistical uncertainty is unavoidable in the process of selecting the distribution $f_X(X)$. The selected $f_X(X)$ may differ from the actual one, causing substantial inaccuracy in the result. The statistical uncertainty is epistemic (knowledge-based) in nature [1], and generally can be reduced if more data is available. However, comprehensive data acquisition may be costly, thus not always economically justifiable. In many practical cases, there is the question of what is the effect of the statistical uncertainty on the computed structural reliability.

A common source of statistical uncertainties arises in the estimation of the parameters (e.g., mean, variance) of the distribution function for the basic random variables. As the parameters are estimated by statistical inference from observational data, errors of estimation are unavoidable when the available data are limited. This error is referred to as *parameter uncertainty*. It is this type of epistemic uncertainty that the present paper addresses.

From a Bayesian point of view, the unknown parameters of a probabilistic model can be modeled by (Bayesian) random variables and introduced formally in reliability assessment. One can compute the expectation of the limit state probability to characterize the total uncertainty, or evaluate the distribution or variance of the limit state probability to separate the effects of aleatory and epistemic uncertainty [1–4]. Judgmental information is needed to estimate the parameters. The estimation of the parameter can be improved by using the Bayesian updating rule when more data become available. However, before receiving additional data the Bayesian approach remains a subjective representation of parameter uncertainty.

Alternatively, the imperfect knowledge about distribution parameter can be expressed by an interval estimation of the

* Tel.: +61 2 93513923; fax: +61 2 93513343.
E-mail address: hao.zhang@sydney.edu.au

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parameter. The method of confidence interval has long been used to construct interval bounds where the unknown distribution parameter is located with a specific degree of accuracy [5]. A probability distribution with interval parameters can be viewed as an *imprecise probability*. Although the interval approach is conceptually simple and arguably requires less subjective information than the Bayesian approach, it is not straightforward to incorporate the effect of interval parameters in reliability analysis. One needs to consider the *families* of all candidate probability distributions whose parameters are within the interval bounds. A practical way to represent the probability family is to specify its lower and upper bounds. As a consequence, the limit state probability will not be unique, but vary between a lower and an upper bound.

Among the various mathematical models using the probability bounding strategy to model an imprecise probability, the probability box (*p*-box in short) method is particularly suitable for representing probability distributions with interval parameters [6]. The probability box method falls within the theories of imprecise probability, and is closely related to other methods that use a similar probability bounding strategy, such as random set and Dempster–Shafer evidence theory [7,8]. For instance, a random set on the real line can be converted to a *p*-box (and vice versa) [6]. For our purpose of reliability assessment, these two methods may be considered to be equivalent. The approach of imprecise probability generally requires less subjective information than the Bayesian approach. It was argued that, from a frequentist point of view, the epistemic uncertainties in the probability distribution can be more faithfully represented using a probability bounding approach [6,9,10].

In 2002, an ''epistemic uncertainty workshop'' sponsored by Sandia National Laboratories was held in Albuquerque, New Mexico [11]. A set of simple test problems involving both epistemic and aleatory uncertainties were solved using different methods, including purely probabilistic approaches, random sets, probability boxes, Dempster–Shafer evidence theory, etc. It was shown that the methods of imprecise probability can provide valuable insight into uncertainty analysis and reliability assessment on the basis of limited data [12]. Schweiger and Peschl [13] employed random sets to describe the variability of material parameters and geometrical data in geotechnical engineering. Random set-valued inputs were used in FE analysis for a deep excavation problem. The vertex method was used to propagate the random set variables through the FE analysis, assuming that the structural response is strictly monotonic with respect to each random set variable. Tonon et al. [14] used random set theory to calculate reliability bounds for an aircraft wing structure. The limit state (wing tip displacement) was known explicitly in terms of a linear function of the random set variables. The Cartesian product method and interval arithmetic was used to propagate the random sets through the analysis. Adduri and Penmetsa [15] considered structural reliability calculation in the presence of both random variables and interval variables. Response surface method was used to approximate the implicit limit state functions as a closed-form expression in terms of the uncertain variables. Oberguggenberger and Fellin [16] used Tchebycheff's inequality to construct random set model (probability box) of a variable using only knowledge of its first and second moments. Two geotechnical applications were presented. The work did not address the issue of propagating random sets (*p*-boxes) in a complex system. Zhang et al. [17] developed a direct interval Monte Carlo sampling method for structural reliability assessment under parameter uncertainties. Probability boxes are used to represent cumulative distribution functions with interval parameters. Interval-valued samples are generated according to the probability boxes. In each simulation, an interval finite element analysis is performed to compute the range of the limit state. After a large number of simulations, bounds of the limit state probability are obtained.

Despite these research effort, practical application of imprecise probability theories in structural reliability assessment is still very difficult. For one thing, the formal propagation of probability boxes (random sets), as presented in [18], uses a Cartesian product method, which can impose significant computational burden. Such difficulty prompted the development of sampling methods for imprecise probability [17,19]. Sampling methods also have practical advantages over the Cartesian product method since they do not require discretizations of continuous unbounded *p*-boxes, thus avoid the discretization errors and loss of accuracy due to truncation of the distribution tails which are of greatest concern in reliability assessment. However, being a sampling method, direct interval Monte Carlo requires a high number of samples to control the sampling error. Thus the total computational cost can still be very costly as each simulation may involve an expensive interval analysis. This highlights the need to investigate the applicability of efficient sampling techniques, such as importance sampling, to imprecise probability.

Furthermore, existing methods for analysis with imprecise probability all require to compute the range (or bounds with reasonable accuracy) of the limit states knowing that the basic variables vary in certain intervals (this is discussed in some detail in Section 2). This is a typical interval analysis problem, and relatively easy if a closed-form expression for the limit state is available. For most reliability analyses of structures of practical interest, the structural responses defined as the limit states are determined implicitly through FE analysis. In particular, nonlinear FE analysis is increasingly being used in the structural analysis part of reliability calculation as it is more capable of capturing the limit state strength and stability of a real structure [20,21]. Thus the reliability assessment with imprecise probability requires multiple interval FE analyses.

Interval FE analysis itself is a very challenging task. Reviews of interval FE analysis can be found in [22,23]. Although reliable *linear elastic* interval FE methods have been developed (e.g., see [23–28]), their computing cost is generally considerably higher than the corresponding deterministic FE methods. Also, the implementation of interval FE analysis is often intrusive, in the sense that existing deterministic FE analysis code cannot be utilized. Perhaps a more severe limitation is that efficient and practical methods for *nonlinear* interval FE analysis are not as yet available at present. The global optimization strategy has been suggested to perform nonlinear interval FE analysis [22,29]. This approach, however, can be very expensive and is only practical for rather simple problems. Another approach is to use sampling method, i.e., sample from the intervals of input parameters, which can be converted to uniform distributions, with hope that the samples will fall sufficiently close to the values giving extremal system responses. The sampling method requires a very large number of samples to obtain a good approximation for the actual response range. The complexity of performing interval FE analysis, particularly for nonlinear structures, further compound the difficulties in applying the theories of imprecise probability to structural reliability assessment. Note that the burden of interval structural analysis may be reduced if the limit state is strictly monotonic with respect to the input variables. The monotonicity assumption, however, is generally difficult to verify rigorously, even for linear elastic problems.

In order to overcome the above two difficulties, this paper develops a new interval importance sampling method for structural reliability calculation under parameter uncertainties. The proposed method is an improvement of the direct interval Monte Carlo method as presented in [17]. The new computation procedure does not require interval FE analyses, and utilizes existing deterministic FE analysis code.

The concept of probability box and direct interval Monte Carlo method is described first. After that, the present formulation is