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Converting reliability constraints by adaptive quantile estimation

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ABSTRACT

The safety factor required to achieve certain reliability turns out to be related to the quantile of the normalized performance index of interest. Quantile functions (quantiles as functions of design parameters) are therefore essential to convert a reliability constraint into the equivalent safety-factor constraint. It is shown in this paper that the estimation of these quantile functions can be achieved by fitting the tail of the normalized performance index. In the cases where the tail varies drastically with the design parameters, a heuristic algorithm is developed to find a series of probability distributions to adaptively fit the tails. Once these probability distributions are obtained, a series of quantile functions can be found to facilitate the conversion of the reliability constraint. Three examples are investigated to verify the proposed approach. Although the theoretical bounds for the approach cannot be proved, the results show that the approach can effectively convert reliability constraints of the three examples into equivalent safety-factor constraints.

1. Introduction

Reliability-based optimization (RBO) [1–5] has recently become an important research area because of the need of making decisions under uncertainties in engineering applications. One of the difficulties encountered in RBO is related to the reliability constraints, to directly ensure which during the optimization algorithm may require numerous reliability analyses. The required computational cost can be unacceptable, rendering many realistic RBO problems computationally intractable. One possible solution is to convert these reliability constraints into non-probabilistic ones by first estimating failure probability as a function of the design parameters. This approach was taken in Gasser and Schüeller [2] and Jensen [5], where the logarithm of such a function is assumed to be either linear or quadratic in the design parameters. The similar approach was also taken with response surface methods or surrogate-based methods [6,7].

1.1. Connection between required safety factor and target failure probability

Ching [8] proposed a novel approach to convert reliability constraints into non-probabilistic ones by using an equivalence theorem between reliability and safety factor. He showed that a reliability constraint can be converted into a safety-factor constraint, and the required safety factor is exactly the $1-P_F^*$ quantile of the "normalized" performance index (P_F^* is the target failure probability). What follows reviews his findings. Let $Z \in \boldsymbol{R}^{\boldsymbol{p}}$ be the uncertain variables of the target system and $\theta \in \boldsymbol{R}^{\boldsymbol{q}}$ be the design parameters; F denotes the failure event: $F \equiv \{R[Z,\theta] > 1\}$, where $R[Z,\theta]$ is called the performance index; $\bar{R}(\theta)$ is a "nominal" performance index: an example of $\bar{R}(\theta)$ is to take $R[Z,\theta]$ but fix Z at certain nominal values. The safety factor approach of design is to enforce the following constraint:

$$\eta^*(\theta)\cdot\bar{R}(\theta) \leqslant 1 \quad (1)$$

where $\eta^*(\theta)$ is the required safety factor; in general, it may depend on θ. On the other hand, the reliability-based design approach is to enforce the following constraint during the design process:

$$P(R[Z,\theta] > 1|\theta) \leqslant P_F^* \quad (2)$$

A theorem developed in [8] states that the two constraints in (1) and (2) are equivalent if the safety factor $\eta^*(\theta)$ is found by solving the following relation:

$$P(G(Z,\theta) > \eta^*(\theta)|\theta) = P_F^* \quad (3)$$

where $G(Z,\theta) = R[Z,\theta]/\bar{R}(\theta)$ is the "normalized" performance index. Note that $\eta^*(\theta)$ is simply the $1-P_F^*$ quantile of $G(Z,\theta)$. The proof of this theorem can be found in [8]. Therefore, finding the required safety factor corresponding to a certain target failure probability is equivalent to finding a quantile of $G(Z,\theta)$.

The aforementioned theorem is practical only when $\eta^*(\theta)$ does not vary with θ, or equivalently, when the quantiles and hence the distribution of $G(Z,\theta)$ do not vary with θ; otherwise, the problem of determining a θ-dependent quantile $\eta^*(\theta)$ may be just as

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difficult as the original RBO problem. In the case where $\eta^*(\theta)$ is a constant η^*, the $\eta^* - P_F^*$ relation can be found by the following equation:

$$P(G(Z,\theta) > \eta^*) = P_F^* \tag{4}$$

Note that θ has been removed from the condition since the conditional probability does not depend on θ. In fact, θ can be fictitiously treated as random and uniformly distributed over a prescribed allowable design region. Fig. 1 illustrates a scenario where the quantiles of $G(Z, \theta)$ do not vary with θ. Note that the G level corresponding to the p-quantile in the figure is exactly the required safety factor corresponding to a target failure probability of $1 - p$.

The key to the success of the theorem developed in [8] is a proper choice of the nominal performance function $\overline{R}(\theta)$: such a proper choice can make the distribution (or quantiles) of $G(Z,\theta) = R[Z,\theta]/\overline{R}(\theta)$ invariant over θ. In [8], it is argued that finding a nominal function $\overline{R}(\theta)$ such that the distribution of $G(Z, \theta)$ is roughly invariant over θ is usually not a difficult task: both $\overline{R}(\theta) = R[E(Z),\theta]$ or $\overline{R}(\theta) = E_Z(R[Z,\theta])$ are conjectured to be possible choices. This is because the distribution of $R[Z, \theta]/R[E(Z), \theta]$ or $R[Z, \theta]/E_Z(R[Z, \theta])$ might not vary drastically with θ due to the possible cancellation effect between $R[Z, \theta]$ and $R[E(Z), \theta]$ (or $E_Z(R[Z, \theta])$).

1.2. Difficult cases where the quantiles are not constant

However, there are cases where the above two choices of $\overline{R}(\theta)$ are not proper, i.e. the quantiles of the resulting $G(Z, \theta)$ vary significantly with θ. See Fig. 2 for such a scenario. For these cases, the required safety factor to achieve a target failure probability P_F^* would change with the design scenario θ, rendering the theorem not practical.

A slight modification of the theorem may resolve the aforementioned issue. Suppose the distribution of $G(Z, \theta)$ varies with θ, but let us assume there exists a monotonically increasing mapping L_θ parameterized by θ such that the distribution of $L_\theta[G(Z, \theta)]$ is invariant over θ, i.e. such a L_θ mapping somehow counteracts the effect of $G(Z, \theta)$. Under this assumption, $\eta^*(\theta)$ will not be a constant but $L_\theta[\eta^*(\theta)]$ will be. This can be easily seen from the fact that

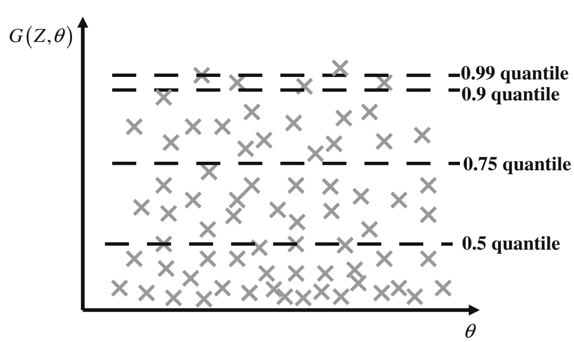

Fig. 1. A scenario where the quantiles of $G(Z, \theta)$ do not vary with θ.

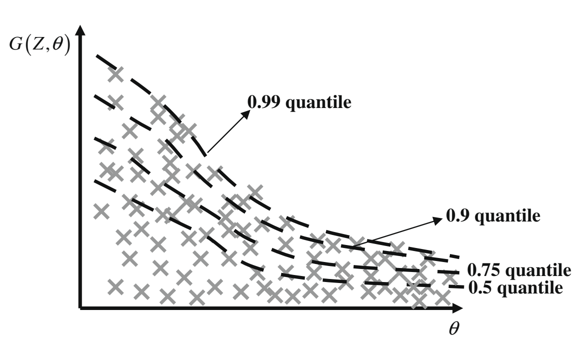

Fig. 2. A scenario where the quantiles of $G(Z, \theta)$ vary with θ.

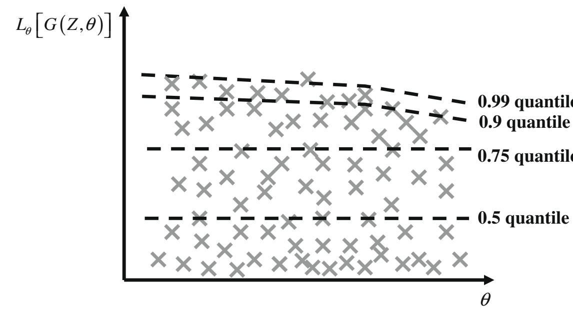

Fig. 3. A scenario with highly variant $G(Z, \theta)$.

$$P_F^* = P(G(Z,\theta) > \eta^*(\theta)|\theta) = P(L_\theta[G(Z,\theta)] > L_\theta[\eta^*(\theta)]|\theta) \tag{5}$$

The last conditional probability term implies that $L_\theta[\eta^*(\theta)]$ must be a constant λ^*. The λ^* value corresponding to a target failure probability P_F^* can then be found by solving

$$P(L_\theta[G(Z,\theta)] > \lambda^*) = P_F^* \tag{6}$$

where θ has been removed from the condition and treated as random and uniformly distributed over the prescribed allowable design region. Once λ^* is found, the required safety factor is simply $L_\theta^{-1}(\lambda^*)$. Therefore, it is not necessary to solve $\eta^*(\theta)$ for each design scenario θ but only necessary to solve for the constant λ^*.

It is proposed in [9] to take L_θ as the estimated cumulative density function (CDF) of the tail of $G(Z, \theta)$. This choice works because any random variable after being transformed by its CDF will be uniformly distributed over the [0, 1] interval. Therefore, under this choice, the tail of $L_\theta[G(Z, \theta)]$ will be roughly uniformly distributed over [0, 1] regardless the value of θ, hence the high quantiles of $L_\theta[G(Z, \theta)]$ are roughly invariant over θ.

1.3. Focus of this study

For difficult cases, it is found that a single L_θ function is usually not enough to ensure the $L_\theta[G(Z, \theta)]$ distribution to be invariant over its entire tail region. Fig. 3 shows a scenario where the application of a single L_θ makes the low quantiles of $L_\theta[G(Z, \theta)]$ relatively constant but the high quantiles of $L_\theta[G(Z, \theta)]$ can still vary with θ. This can happen, for instance, in a problem with switching failure modes. Nonetheless, it is found that a series of $L_\theta^1, L_\theta^2, \ldots, L_\theta^m$ mappings can be more effective. This implies that solving a sequence of $\eta_1^*(\theta) < \ldots < \eta_m^*(\theta)$ that approach $\eta^*(\theta)$ may be possible, and this is the main focus of this paper.

2. Reliability constraints

Given the design parameters θ, the probability of failure of the target system is

$$P(F|\theta) = \int_{\Omega_{F|\theta}} p(z|\theta)dz \tag{7}$$

where $p(z|\theta)$ is the probability density function (PDF) of Z; $\Omega_{F|\theta}$ is the failure domain in the $\boldsymbol{R^p}$ space: $\Omega_{F|\theta} = \{z{:}R[z, \theta] > 1\}$. The performance index $R[Z, \theta]$ does not necessarily define the complete collapse of the system but the performance of the system, e.g. serviceability and ultimate capacity. Throughout the paper, it is assumed without loss of generality that $R[Z, \theta]$ is positive and that