Structures 4 (2015) 2–12

Contents lists available at ScienceDirect

Structures

journal homepage: http://www.elsevier.com/locate/structures

# A new approach to modal decomposition of buckled shapes

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ARTICLE INFO

*Article history:*
Received 30 June 2015
Received in revised form 22 October 2015
Accepted 22 October 2015
Available online 30 October 2015

*Keywords:*
Buckling modes
Classification
Decomposition
Local buckling
Distortional buckling
Instability

ABSTRACT

The paper proposes a novel methodology to construct the individual local, distortional, and global buckling modes of a thin-walled structural element under a given loading. The resulting buckling modes form an orthogonal basis of the deformation space, so that any random deformed shape can be expressed as a linear combination of the basic buckling modes. The method is applicable to open branched or unbranched cross sections, as well as cross sections containing closed parts. Examples are provided to illustrate the procedure.


## 1. Introduction

Thin-walled structures are used in many applications across a range of engineering disciplines, including structural, aeronautical, nautical, and mechanical engineering. They offer many advantages, most notably an efficient use of material and, consequently, a high strength-to-weight ratio. However, owing to their limited wall thickness, they are typically susceptible to a number of buckling modes, and the study of their stability offers an interesting and challenging field of study. Quite often, various buckling modes may interact with each other to produce a detrimental effect on the load bearing capacity through mechanisms which are often sensitive to initial geometric imperfections. In order to study and understand these complex phenomena, it is useful to be able to separate coupled instabilities into a number of 'standard' individual modes of which the characteristics and behavior are relatively well known. Traditionally, buckling modes in thin-walled structural elements are classified into local, distortional, and global modes. Fig. 1 illustrates some of these modes for the case of a lipped channel under compression. Local, distortional, and global modes differ significantly in terms of their behavior, particularly with respect to the post-buckling capacity they display. The local modes typically have ample post-buckling reserve capacity when buckling in the elastic range, while distortional buckling is associated with significantly less post-buckling capacity and the global modes possess virtually none. Identifying and classifying buckling modes is therefore an important component of assessing the behavior and capacity of thin-walled structural elements.

The general aims of this paper are (1) to propose a methodology to construct the individual local, distortional, and global buckling modes of any specific thin-walled element under a given loading, and (2) to provide a means to identify the contributions of these 'pure' modes in a random deformed shape, which may be a coupled instability. It is thereby noted that several tools for the stability analysis of thin-walled elements currently exist, the most commonly used ones being the finite strip method (FSM) [10] and the more general finite element method (FEM). Both are able to perform a buckling analysis, however, they return the eigenvalues (load parameters) associated with modes which are often coupled instabilities and not the 'pure' local, distortional, or global modes we wish to obtain as the aim of this research.

It should also be noted that, as previously established (e.g., by [7, 12]), a complete description of all possible buckling modes also necessitates the introduction of shear modes and transverse extension modes in addition to the traditional local, distortional, and global modes.

Previous attempts at modal classification and decomposition have mainly centered around Generalized Beam Theory (GBT), originally developed by Schardt [19]. GBT was first developed for unbranched open sections and simple closed sections and later extended to include more general sections [13] and even further developed for some curved cross-sectional shapes [20]. GBT provides a way to arrive at the pure 'deformation modes' by uncoupling the first-order equilibrium differential equations. It should also be noted that, as an extension of classical beam theory, GBT is based on certain idealized assumptions. The

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http://dx.doi.org/10.1016/j.istruc.2015.10.011
2352-0124/


**Fig. 1.** Buckling modes of a lipped channel: a. local, b. distortional, and c. global modes.

success of GBT in achieving pure uncoupled modes, however, has led to the development of the constrained finite strip method (cFSM), whereby the principles and the mechanics of GBT are introduced into the FSM in order to obtain a modal decomposition. Early work on the cFSM was carried out by Adany [1], followed by the presentation of a formal derivation [5] and a companion paper with applications and examples [6]. At that stage, however, the method could only be applied to single-branched, open cross sections. A study into shear modes in thin-walled members [3] later paved the way to a generalized cFSM for arbitrary cross sections [8,9], with an alternative generalization developed by Djafour et al. [11]. An extension of the cFSM to include general end boundary conditions was also presented by Li and Schafer [14]. Adany et al. [4] extended the range of application of the cFSM by using the method to identify finite element generated buckling modes in thin-walled members. Research with a similar purpose was carried out by Nedelcu [16,17] and Nedelcu and Cucu [18], who used GBT to decompose the elastic buckling modes obtained from shell finite element analysis, with or without perforations.

## 2. Objectives

The general aims of the research are to develop a methodology

1. to create a set of pure buckling modes for a given thin-walled structural element under a given loading (e.g., compression or bending), categorized into local, distortional, global, and 'other' modes (i.e., shear and transverse extension modes).
2. to decompose a random deformed shape into a linear combination of the pure buckling modes.

In addition, the set of pure buckling modes should satisfy the following criteria:

1. The full set of local, distortional, global, and other modes form an orthogonal basis of the full deformation space. Orthogonality is thereby expressed with respect to the elastic stiffness matrix **K**. This is a logical choice since an elastic buckling analysis (e.g., using the FSM) requires the solution of an eigenvalue problem of the form:

$$(\mathbf{K}-\lambda\mathbf{G}).\mathbf{v}=0 \tag{1}$$

where **K** is the elastic stiffness matrix and **G** is the geometric stiffness matrix (or stability matrix). The resulting eigenvectors **v** define the buckled shapes, while the associated eigenvalues λ are proportional to the elastic buckling stresses. It is easily proven that all eigenvectors **v** are orthogonal to each other with respect to **K**, as well as with respect to **G.** Indeed, let **v**i and **v**j be two eigenvectors associated with two different eigenvalues λi and λj, then

$$(\mathbf{K}-\lambda_i\mathbf{G}).\mathbf{v}_i=0 \tag{2}$$

$$(\mathbf{K}-\lambda_j\mathbf{G}).\mathbf{v}_j=0 \tag{3}$$

Pre-multiplying Eq. (2) by $\mathbf{v}_j^T$, pre-multiplying Eq. (3) by $\mathbf{v}_i^T$, and subsequently transposing Eq. (3) leads to:

$$\mathbf{v}_j^T.(\mathbf{K}-\lambda_i\mathbf{G}).\mathbf{v}_i=0 \tag{4}$$

$$\mathbf{v}_j^T.\left(\mathbf{K}^T-\lambda_j\mathbf{G}^T\right).\mathbf{v}_i=0 \tag{5}$$

Subtracting Eq. (5) from Eq. (4) and using the symmetry of K and G leads to:

$$(\lambda_i-\lambda_j)\ \mathbf{v}_j^T.\mathbf{G}.\mathbf{v}_i=0 \tag{6}$$

Since $\lambda_i$ and $\lambda_j$ are assumed to be different eigenvalues, Eq. (6) proves the orthogonality of the eigenvectors with respect to **G**,

$$\mathbf{v}_i^T.\mathbf{G}.\mathbf{v}_j=0 \quad \forall i\neq j \tag{7}$$

and consequently, through Eq. (4) or Eq. (5), with respect to **K**.:

$$\mathbf{v}_i^T.\mathbf{K}.\mathbf{v}_j=0\forall i\neq j \tag{8}$$

Orthogonality with respect to **K** has the physical meaning that the work done by the stresses associated with $\mathbf{v}_i$ in the strains associated with $\mathbf{v}_j$ is zero (and vice versa). This inspires us to define an inner product over the space of possible deformations whereby the inner product of two vectors **v** and **w** is given by:

$$<\mathbf{v}|\mathbf{w}>=\frac{1}{2}\mathbf{v}^T.\mathbf{K}.\mathbf{w}=\frac{1}{2}\mathbf{w}^T.\mathbf{K}.\mathbf{v} \tag{9}$$

Compared to Eq. (8), a factor of ½ has been added, so that the inner product has the physical meaning of elastic strain energy.

Although the derivation of the proposed methodology does not rely on it, it is worth noting that if the complete set of pure buckling modes maintains orthogonality with respect to **K**, a property also encountered in the solutions of the stability problem Eq. (1), then there necessarily exists a rotation in 4 N-dimensional space (i.e., the space spanned by all degrees of freedom: N is the number of nodes in the FSM model) which rotates the output of Eq. (1) into the pure modes. This is true provided that all modes are normalized and that the correct orientation of the vectors is chosen along their axes (+1 or −1; it is obvious that this choice is completely arbitrary). Since the inner product, through Eq. (4), is defined with respect to **K**, the logical choice within the context of this paper is to also normalize all eigenmodes with respect to **K**, so that:

$$<\mathbf{v}_i|\mathbf{v}_i>\ =\ \frac{1}{2}\mathbf{v}_i^T.\mathbf{K}.\mathbf{v}_i=1 \tag{10}$$

With the aim in mind of decomposing a random deformed shape into its constituent pure modes, it is obviously necessary that the buckling modes form a set of independent basis vectors of the deformation space. Imposing orthogonality on top of this requirement has the advantage that the decomposition of a random shape can be achieved by simply projecting the shape onto the basis vectors using the inner product.

2. The decomposition method presented by Adany and Schafer [7], the cFSM, borrows from the principles of Generalized Beam Theory (GBT) to distinguish between local, distortional, and global buckling.