Thin-Walled Structures 94 (2015) 293–301

Contents lists available at ScienceDirect

Thin-Walled Structures

journal homepage: www.elsevier.com/locate/tws

# On the methods of critical load estimation of spherical circle axially symmetrical shells

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ARTICLE INFO

*Article history:*
Received 17 June 2014
Received in revised form
30 March 2015
Accepted 2 April 2015
Available online 14 May 2015

*Keywords:*
Stability
Shells
Critical loads
Chebyshev's method

ABSTRACT

A relaxation method is applied to estimate and predict a critical set of parameters responsible for stability loss (buckling) of spherical circle axially symmetric shells. The buckling phenomenon under static loading was investigated by solving the Cauchy problem for a set of ordinary differential equations and the Hausdorff metrics was applied while quantifying the data obtained within the novel approach.


## 1. Introduction

One of the key issues in the field of materials strength and structural mechanics is that devoted to the study of stability as well as buckling and postbuckling behavior of structural members such as beams, plates, shells and thin-walled structures. These members of structures and the structures themselves are usually subjected either to static or dynamic, or both types of loading, and hence various computational techniques, including numerical, analytical and combined numerical–analytical approaches are used to analyze various types of stability loss including local, global (flexural, torsional, lateral, distorsional and their combinations) and interactive forms of buckling.

It is well known that the mentioned either isolated or coupled structural members have found wide applications in numerous constructions in aerospace, civil engineering, ship building, automobiles, aircraft wings and fuselages, and others. There are numerous papers/monographs devoted to stability loss (and buckling) investigation of structural members treated as isolated or interacting objects, where the structural members are linked with each other by different/mixed boundary conditions. It is well recognized that stability loss is understood as the transition of a mechanical system from one to another equilibrium configuration either in a smooth way (bifurcation point) or by a sudden jump from a stable to unstable equilibrium path (limit point).

In general, there are either static or dynamic loads. The latter ones are measured via “pulse intensity” and “pulse velocity”. Depending on their length in time, different dynamic loading phenomena can be distinguished. Namely, when pulse duration is short (long) and the amplitude is relatively high (average) then an impact (quasi-static) behavior is observed. In the case when the pulse duration is close to the period of natural vibrations, a dynamic buckling takes place.

It should be emphasized that a finite duration load may have different shapes (parabolic, triangular, rectangular, exponential or even irregular), since it attempts to model real dynamic load met in nature and engineering applications. Studies on the stability and buckling behavior of mainly thin-walled structures date back to over a hundred years, and were motivated by Bernoulli/Euler [1], Timoshenko [2] and Volmir [3,4]. Here, our studies are limited to only a few proposals regarding stability phenomena, but the reader may find more information for instance in the recent monograph of Kubiak [5].

Growing interest in stability loss/buckling/postbuckling behavior of thin-walled structures measured by the publication of a number of papers/books began in the 1970s (see for instance [6–12]). In particular, a lot of research was aimed at non-linear problems of stability of orthotropic and anisotropic thin-walled structures. The studies covered orthotropic plate buckling [13], critical stresses of anisotropic laminated plates [14], buckling of composite and anisotropic plates [15–17], stability of columns and square laminate plates [18] and postbuckling behavior of

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http://dx.doi.org/10.1016/j.tws.2015.04.003
0263-8231/

orthotropic laminated plates [19]. Numerous papers have been devoted to the solution of stability problem using numerical and analytical–numerical methods often applying commercial programs based on the finite elements method (FEM). However, despite the mentioned research aiming at the explanation of the static/dynamic stability loss of thin-walled structures and structural members, there is no a general stability definition/criterion formulated which can be validated experimentally and can satisfy engineering requirements regarding the load carrying capacity as well as stability of the mentioned mechanical objects. Below, a few of stability loss criteria regarding continuous mechanical objects which found considerable resonance among researchers are briefly illustrated and discussed.

Volmir [4] proposed a time-consuming though simple approach to determine the critical load while investigating dynamics of a simply supported rectangular plate subjected to pulses of infinite/finite durations and of rectangular/exponential shapes. He pointed out that the plate subjected to pulse load lost its stability, when the maximum deflection of the plate was equal to the assumed constant value (usually, it was either the plate thickness or the half-plate thickness). Budiansky and Hutchinson [20,21] and Budiansky and Roth [22] proposed displacement criteria regarding cylindrical shells axially loaded rods and cylindrical shells loaded transversally, respectively. There are two equivalent formulations of their criteria: 1) structures subjected to pulse loading lose their stability when an unlimited increase of their deflection for small load increments is observed; 2) a plate exhibits dynamic stability loss when its maximum deflection grows rapidly under a small load amplitude variation.

Both theoretical and experimental investigations of thin plates clamped on all contours and subjected to pulse load with a half-wave of sine shape carried out by Ari-Gur and Simonetta [23] yielded other four dynamic criteria. Only two of them are recalled (the other two deal with failures): 1) dynamic buckling takes place when a small increase of the loading pulse intensity causes a significant increase of the deflection value; 2) dynamic buckling takes place when a small increase of the pulse loading amplitude causes a decrease of the deflection value.

The so far discussed stability loss criteria concern isolated structural members like beams, columns, plates and shells. In complex structures composed of the linked simple structural members the problem is more difficult. One buckling mode may simply create other modes, and then a problem of multi-modal modes stability appears. Petry and Fahlbusch [24] extended the Budiansky–Hutchinson criterion to plated structures, and they proposed the following dynamic buckling criterion: *Dynamic response of a structure subjected to pulse load is dynamically stable if the condition that the equivalent stress* (originally the authors used the Huber–Mises hypothesis) *less/equal to the assumed limit of stress is satisfied at any time and any point of the structure*.

In addition to the presented status of existing criteria of the structural members stability loss, a few papers from the Russian literature are referred to. Kulikov [25] studied the stability of a spherical shell putting emphasis on numerical techniques applied to study non-linear behavior of thin elastic shells. Numerous algorithms of the FDM (Finite Difference Method) devoted to the solution of stability problems of mechanical structures allow researchers to solve a large class of static and dynamic problems. Valishvili [26] solved the static problem, where the non-linear boundary value problem was reduced to that of solving non-linear algebraic equations. In addition, static problems of structural membranes can be solved with the help of a relaxation method first applied by Feodos'ev for shells [27].

The so far given review of papers devoted to stability/buckling problems of structural members shows that there are numerous approaches to define and predict this phenomenon. However, it is also clear that none of them is sufficient and meets expectations of the engineering community. In general, models of the processes associated with stability loss of mechanical structures require derivation of complex variational equations or equivalent differential equations. Additionally, in spite of a large number of algorithms devoted to the computation of various kinds of stability loss and in spite of the used characteristics such as graphical stability loss visualization versus the applied load, there is no relatively simple and reliable estimation of stability loss pictures being validated by various laboratory experiments.

The aim of this paper is to get reliable characteristics of time evolution of the development of shell deformation versus the applied load variation in order to detect the critical load values. For this purpose the following problems are solved: (i) to estimate the deformation velocity while changing an input load; (ii) to get information on rapid changes of the deformation velocity development in order to reach a certain critical load level; (iii) to get information on the absolute and relative error introduced by the linear approximating function while changing the initial input. It should be emphasized the static problems are solved by using the dynamic method which is much more efficient in comparison to the standard static approaches.

The paper is organized in the following manner. In Section 2 both the method and algorithm of computation of an axially symmetric spherical shell are presented. Section 3 deals with a static stability loss of shells. A core of the paper is in Section 4 devoted to the application of modified Chebyshev's method used to quantify the velocity characteristics of shell deflection. In particular, an important theorem is formulated. Section 5 presents computational experiments validating the previous theoretical considerations. Concluding remarks sum up the research carried out and the novel results obtained.

## 2. Method and algorithm of computation

Consider a shallow spherical axially symmetrical shell described by the 2D space in $R^2$ in the polar co-ordinates introduced in the following way: $\Omega = \{(r,z) \mid r \in [0,b], \ -H/2 \le z \le H/2\}$.

Dynamics of the mentioned axially symmetric shells is governed by the following set of PDEs:

$$w'' + \varepsilon w' = -\frac{\partial^4 w}{\partial r^4} - \frac{2}{r}\frac{\partial^3 w}{\partial r^3} + \frac{1}{r^2}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r^3}\frac{\partial w}{\partial r} - \frac{\Phi}{r}\left(1 - \frac{\partial^2 w}{\partial r^2}\right) - \frac{\partial \Phi}{\partial r}\left(1 - \frac{1}{r}\frac{\partial w}{\partial r}\right) - 4q, \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} - \frac{1}{r^2}\frac{\partial \Phi}{\partial r} = \frac{\partial w}{\partial r}\left(1 - \frac{1}{2r}\frac{\partial w}{\partial r}\right), \tag{2.1}$$

where $\Phi = \partial F/\partial r$ and $F$ stands for the stress (Airy's) function. While investigating theoretically real shells, usually a 3D problem of the theory of elasticity is reduced to that of 2D assuming that the shell material is elastic and satisfies Hook's law and that the Kirchhoff–Love hypothesis is validated (normals to the middle shell surface are not deformed with shell deformation).

System (2.1) is recast into its counterpart dimensionless form by introducing the following relations:

$$\bar{t} = \omega_0 t, \ \ \omega_0 = \sqrt{\tfrac{Eg}{\gamma R^2}}, \ \ \bar{\varepsilon} = \sqrt{\tfrac{g}{\gamma E H}}\tfrac{R}{}\varepsilon, \ \ \bar{F} = \eta \tfrac{F}{EH^3}, \ \ \bar{w} = \sqrt{\eta}\tfrac{w}{H}, \ \ \bar{r} = b\tfrac{r}{c},$$
$$\bar{q} = \tfrac{\sqrt{\eta}}{4}\tfrac{q}{E}\left(\tfrac{R}{H}\right)^2, \ \ \eta = 12(1-\nu^2), \ \ b = \sqrt[4]{\eta}\tfrac{c}{\sqrt{RH}},$$

where $t$ – time, $\varepsilon$ – damping coefficient, $F$ – stress function, $w$ – displacement function, $R$ – main shell curvature radius, $2c$ – length of ends of the shell curvature (see Fig. 1), $H=2h$ – shell thickness (see Fig. 1), $b$ – shallow parameter, $\nu$ – Poisson's coefficient, $r$– distance between the axis of rotation and a point of the middle