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Linearly elastic constitutive relations and consistency for GBT-based thin-walled beams

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ABSTRACT

The present paper focuses on the constitutive assumptions, both for the isotropic and orthotropic cases, and consistency in the framework of the Generalized Beam Theory. In particular, a novel approach based on energetic arguments, able to automatically select appropriate constitutive relations in accordance with the GBT kinematics, is presented. Furthermore, the concept of consistency of a GBT-based model is established and a consistency analysis is presented. This yields a formal rational basis to investigate the effects of the various families of cross-section deformation modes in terms of predictive capabilities of the GBT model. Some numerical examples illustrate the arguments exposed in the paper.

1. Introduction

Thanks to their high stiffness and reduced self-weight, thin-walled beams are nowadays used in a broad variety of applications ranging form aeronautical to civil engineering. Due to the in-plane cross-section deformability and their complex warping behaviour, classical beam theories cannot be in general deemed valid and three-dimensional shell models or enriched beam models must be adopted for their analysis.

Moving from the well-known theory developed by Vlasov [1], a large amount of theoretical work has been proposed in the literature aiming at defining mono-dimensional formulations which are able to reproduce the complex three-dimensional behaviour of thin-walled beams. In particular, Capurso [2–4] extended the Vlasov theory by enriching the warping description while keeping null in-plane deformation of the cross-section. Such an approach has been later further developed by many authors, introducing the concept of generalized warping functions and taking into account also the in-plane cross-section deformability (some recent examples can be found in [5,6]). In this respect, the Generalized Beam Theory (GBT), originally proposed by Schardt [7,8] in the 1980s, has been proven to be able to account for cross-section distortion along with the more "classical" kinematics of axial displacement, bending and torsional rotation in a comprehensive fashion.

Following the work of Schardt, many authors have contributed to the improvement of the GBT by extending it beyond its original formulation for open unbranched sections [9–12], by adding geometric and material nonlinear effects [13–18], by developing beam elements based on semi-analytical solutions [19], or by presenting new approaches for the selection of the cross-section modes [20–22]. Moreover, an interesting application of the GBT to analyse cold-formed roof systems has been presented in [23], and an effective equilibrium-based procedure for the reconstruction of the three-dimensional stresses in GBT members has been proposed in [24]. In the GBT literature, much attention has also been devoted to the shear deformability [13,25–27]. In particular, a new formulation of the GBT that coherently accounts for the shear deformation has been recently presented in [27] and its relationship with classical and non-classical beam theories has been discussed in [28].

Independent of the specific variant, the fundamental idea of the GBT remains to assume the displacement field of the beam to be a linear combination of predefined cross-section deformation modes (which are selected beforehand) multiplied by unknown functions dependent on the beam axial coordinate, called generalized displacements. In this sense, the GBT model can be viewed as a one-dimensional model deduced from a parent three-dimensional one by the introduction a kinematic ansatz.

Of course, depending on the kinematic ansatz, this can lead to a poor (or even null) representation of the three-dimensional strain components over the cross-section (i.e. in the co-dimension of the model) and, in turn, to an over-stiffening which limits the predictive capabilities of the beam model. Such behaviour has been well documented by Silva et al. in various papers, see [16,29,30]. In particular, in [16], it has been shown that the buckling load in some cases can be dramatically overestimated (up to 300%), by using improperly derived GBT finite elements. Indeed, this is a typical problem of structural models with constrained kinematics and it is usually dealt with by properly adjusting the constitutive relationship. A remarkable example of this strategy is represented by shear correction factors usually employed in Timoshenko beams and in

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shear-deformable plate models. In the case of non-standard beam models, such an approach leads to some drawbacks. In particular, the value of such coefficients depends on the stress distribution on the cross-section so that, on one side, their value is problem dependent and, on the other, it might change from section to section.

Also in GBT based models the problem is tackled by modifying the constitutive relations, by adopting a different treatment for the membrane part of the strain field (pertaining to the cross-section midline) and for the bending one (outside the cross-section midline). Moreover, in this respect, it should be noticed that in the GBT literature, this different treatment is carried out in a non-univocal way for isotropic and orthotropic beams. As it can be easily argued, the arbitrary adoption of two different constitutive relations for the membrane and the bending parts is not desirable. In fact, this way of proceeding on one side does not give a clear insight on the physical meaning of such an approach and, on the other side, it might lead to non-univocal choices if the displacement field is enriched or laminated beams considered.

In order to overcome these difficulties, an approach able to automatically identify constitutive relations consistent with the adopted kinematic hypotheses is presented in this paper (Section 3). In the proposed approach, constitutive relations are obtained via complementary energy and there is no distinction between membrane and bending parts, nor isotropic and orthotropic materials.

Indeed, the different treatment of the membrane and bending parts, as well as the alternative approach presented in the paper, suffices to overcome the over-stiffening problems in the case of isotropic material, but does not in that of orthotropic material. In this case, in fact, due to the coupling introduced by the constitutive relationship it is necessary to ensure the coherence between the representations over the cross-section of the stress and strain components which, through energetic equivalence, contribute to the definition of the cross-section stiffness matrix. Here, this idea is formalized in a rigorous analysis by means of the concept of consistency. This concept was early introduced by Prathap and his co-workers (see [31] and the references therein) with regard to the assumed displacement finite element model in constrained media elasticity. They showed that consistency offers a conceptual scheme to delineate some well-known deficiencies of the assumed displacement approach and suggests the way to construct variationally correct procedures to overcome these shortcomings [32,33]. Later, the same concept was successfully extended to coupled problems [34–36] and used as formal basis to develop an integrated procedure to recover consistent stresses for displacement based finite elements [37].

In the present paper, the concept of consistency of a GBT model is established and a consistency analysis is proposed (Section 4). The consistency analysis yields a formal rational basis to investigate the effects of the various families of cross-section deformation modes in terms of predictive capabilities of the GBT model. Based on this, the effect of a certain deformation-mode family when the material, the cross-section geometry or the load is changed, can be not only explained but also anticipated a priori. Some numerical examples illustrate the arguments exposed in the paper (Section 5).

2. Generalized beam theory

In this section, the shear-deformable GBT recently proposed in [28] is briefly summarized.

2.1. Kinematics and statics

The displacement field of the beam is assumed to be a linear combination of predefined cross-section deformation modes multiplied by generalized displacements that depend on the beam axial coordinate. Such operation is essentially a variable separation which allows us to map a fully three-dimensional behaviour into its components in the section plane and their distributions in the axial direction. Thus, the following displacement field, $\mathbf{d}$, for the generic i-th wall of the cross-section is assumed (see Fig. 1):

$$\mathbf{d} = \mathbf{U}\mathbf{u}, \tag{1}$$

$$\mathbf{d} = \begin{bmatrix} d_n(s,z,n) \\ d_s(s,z,n) \\ d_z(s,z,n) \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} \boldsymbol{\psi}(s,n) & 0 \\ \boldsymbol{\xi}(s,n) & 0 \\ 0 & \boldsymbol{\omega}(s,n) \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \mathbf{v}(z) \\ \mathbf{w}(z) \end{bmatrix}, \tag{2}$$

where d_n, d_s and d_z are the displacement orthogonal to the wall midline, tangent to the wall midline and in the beam axial direction, respectively, while $\boldsymbol{\psi}$, $\boldsymbol{\xi}$ and $\boldsymbol{\omega}$ are row matrices collecting the assumed cross-section deformation modes (depending only on s and n), and $\mathbf{v}$ and $\mathbf{w}$ are vectors that collect the unknown kinematic parameters (depending only on z). Moreover, cross-section deformation modes $\boldsymbol{\xi}$ and $\boldsymbol{\omega}$ are assumed to depend linearly on n in the form

$$\boldsymbol{\xi}(s,n) = \boldsymbol{\mu}(s) - n\mathring{\boldsymbol{\psi}}(s), \quad \boldsymbol{\omega}(s,n) = \boldsymbol{\phi}(s) - n\boldsymbol{\psi}(s), \tag{3}$$

where $\boldsymbol{\mu}$ and $\boldsymbol{\phi}$ are predefined shape functions. Hereinafter, $\mathring{()}$ and $()'$ denote the derivatives with respect to the s and z coordinates, respectively.

From the above displacement field, it is possible to calculate strains by means of the three-dimensional compatibility equations, yielding $\varepsilon_{nn} = \gamma_{sn} = 0$ and

$$\boldsymbol{\varepsilon} = \mathbf{E}\mathbf{e} \tag{4}$$

where

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{zz} \\ \varepsilon_{ss} \\ \gamma_{zs} \\ \gamma_{zn} \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 0 & \boldsymbol{\phi} - n\boldsymbol{\psi} & 0 & 0 \\ \mathring{\boldsymbol{\mu}} - n\mathring{\mathring{\boldsymbol{\psi}}} & 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{\mu} + \mathring{\boldsymbol{\phi}} - 2n\mathring{\boldsymbol{\phi}} & \frac{1}{2}\left(\boldsymbol{\mu} - \mathring{\boldsymbol{\phi}}\right) \\ 0 & 0 & 0 & \boldsymbol{\psi} \end{bmatrix}, \tag{5}$$

and $\mathbf{e}$ is the vector collecting independent z-fields governing the strain components, denoted as generalized deformation parameters:

$$\mathbf{e}^{\mathrm{T}} = \left[\mathbf{v}^{\mathrm{T}} \ \ \mathbf{w}'^{T} \ \ \tfrac{1}{2}(\mathbf{v}' + \mathbf{w})^{\mathrm{T}} \ \ (\mathbf{v}' - \mathbf{w})^{\mathrm{T}} \right]. \tag{6}$$

For later convenience, the strain field presented in Eq. (4) is split into a membrane part, not depending on n and denoted by $\boldsymbol{\lambda}$, and a bending part, depending on n and denoted by $\boldsymbol{\chi}$, leading to

$$\boldsymbol{\varepsilon} = \boldsymbol{\lambda} + n\boldsymbol{\chi}, \tag{7}$$

where

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_{zz} \\ \lambda_{ss} \\ \lambda_{zs} \\ \lambda_{zn} \end{bmatrix} = \mathbf{E}^{\lambda}\mathbf{e}, \quad \boldsymbol{\chi} = \begin{bmatrix} \chi_{zz} \\ \chi_{ss} \\ \chi_{zs} \\ \chi_{zn} \end{bmatrix} = \mathbf{E}^{\chi}\mathbf{e}, \tag{8}$$

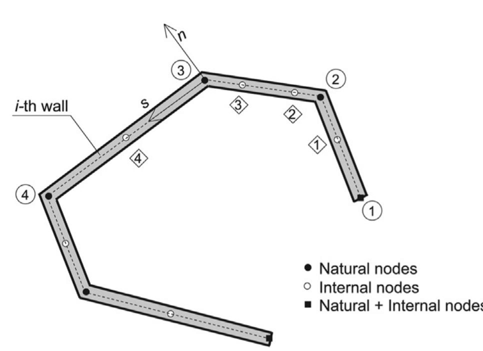

Fig. 1. Thin-walled cross section.