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Equatorial bending of an elliptic toroidal shell

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ABSTRACT

The exact differential equations for the axisymmetric bending of elliptic toroidal shells are difficult to solve. In this paper, and by considering a semi-elliptic toroid, we present an approximate bending solution that is valid in regions adjacent to the horizontal equatorial plane. The formulation accurately simulates edge effects which may arise from loading and geometric discontinuities located in the equatorial plane of elliptic toroids. In particular, the developed closed-form results provide a very effective means for evaluating the state of stress in the relatively narrow zones experiencing mid-side local effects in complete elliptic toroidal vessels subjected to hydrostatic loading, and for calculating the deformed shape of the shell midsurface.

1. Introduction

The membrane theory of shells is a greatly simplified yet very effective basis for estimating stresses and deformations in those regions of the shell over which the loading and geometry do not change too rapidly. However, and as is well-known, the theory becomes inadequate at or in the vicinity of supports, concentrated loadings, shell junctions or discontinuities in shell geometry (thickness, slope, radii of curvature), loading and material properties. Novozhilov [1] has called these locations "lines of distortion" in reference to the existence of a bending effect locally disturbing the membrane state of stress in these regions. Discontinuity problems in shells of revolution have been the subject of many investigations, and a good body of closed-form results exists for the more common types of shells and loading conditions [2–5].

The performance of containment shells is usually assessed with regard to their stress and deformation response in the linear elastic range [2,3], their vibration characteristics and dynamic response, as well as their nonlinear buckling and postbuckling behaviour within the elastic and plastic ranges of material behaviour. Metal shells are particularly susceptible to buckling on account of their thin-ness (radius-to-thickness ratios typically in excess of 500). Numerical studies have been carried out on the buckling capacity of vertical cylindrical steel tanks [6–11], horizontal cylindrical and near-cylindrical vessels [12–14] and conical tanks [15–17]. The buckling capacity of multi-segmented shells under external water pressurisation has also been investigated [18], as has the elastic buckling of certain unusual mathematical forms for shells [19,20].

Junction stresses in various shell assemblies and multi-segmented vessels have been the subject of intensive studies over the past 15 years [21–25]. Mechanics phenomena around shell intersections and at shell-branching locations have also been of interest [26,27]. The presence of ring beams at shell junctions has a considerable influence on the behaviour of the shell; some efforts have also been directed towards understanding ring–shell interactions [28,29]. A more comprehensive review of recent studies on the statics, dynamics and stability of various types of liquid-containment shells under a variety of loading conditions may be seen in a recent survey [30].

Toroidal shells have mostly been studied with pressure-vessel applications in mind, though liquid-containment applications have also been of interest. The classical solutions for pressurised circular and elliptical toroids may be seen in texts on linear shell analysis [2–5]. Even where toroidal shells with uniform geometry are subjected to internal pressure only, the membrane solution becomes inadequate in the vicinity of the horizontal circles furthest from the equatorial plane, owing to the vanishing of the curvature in one of the principal planes [31].

Sutcliffe [32] tackled the stress analysis of both circular and elliptical toroidal shells subjected to internal pressure. While accurate for the purpose, the formulation is somewhat cumbersome

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for practical implementation. Galletly [33] considered the elastic buckling of an elliptic toroidal shell subjected to uniform internal pressure, and confirmed that internally pressurised elliptical toroids, unlike circular toroids, may possibly buckle, depending on the axes ratio of the elliptical cross-section. The study was also extended to plastic buckling [34], for which the post-buckling behaviour of the shell was noted to be stable.

Redekop [35] studied the buckling behaviour of an orthotropic toroidal shell of elliptical cross-section, while Yamada et al. [36] considered the free vibration response of a toroidal shell of elliptic section. Xu and Redekop [37] also considered the free vibration of elliptic toroidal shells, but with orthotropic properties. Zhan and Redekop [38] studied toroidal tanks with cross-sections made up of combinations of circular arcs of different radii (ovaloid shape), and observed the vibration, buckling and collapse behaviour of this type of toroidal vessel.

In this paper, we will focus attention on the thin elliptic toroidal shell. Noting the lack of a convenient analytical solution for the axisymmetric bending of an elliptic toroidal shell, we aim at developing a practical means for estimating bending-disturbance effects that may arise in the mid-side locations (herein referred to as "equatorial" locations) of vertically elongated thin elliptic toroids, where the vertical semi-axis b of the ellipse is greater than the horizontal semi-axis a. Specifically, we aim to develop and present a set of closed-form expressions for interior shell stresses due to axisymmetric bending moments and shearing forces applied in the equatorial plane of the elliptic section as uniformly distributed edge actions.

The formulation is intended for use in quantifying (i) the junction effects in the vicinity of the equatorial plane of subsea elliptic–toroidal shell structures (where a horizontal plate deck may be attached to the inner walls of the toroid to provide an interior working platform extending right round the torus), or (ii) the edge effects in the vicinity of supports where the elliptic toroidal vessel is used as an elevated circular tank supported on closely-spaced vertical columns attached at both the intrados and extrados of the torus. The relatively weak edge effects associated with partial filling of the tank may also be quantified on the basis of this solution. We will begin by defining the geometry of the elliptic toroid.

2. Geometrical preliminaries

Fig. 1 shows the relevant geometrical parameters of an elliptic toroidal shell. To generate the torus, an ellipse of semi-axes a (horizontal) and b (vertical) is rotated about a vertical axis $Y-Y$ that lies at a distance A ($>a$) from the local vertical axis $y-y$ of the ellipse. The equatorial plane (horizontal plane of symmetry) is denoted by $E-E$. In what follows, we will take the generator curve (or meridian) of the toroidal shell as the ellipse to the left of the axis $Y-Y$. Let P be any point on the generator meridian. The radius of curvature of the ellipse at point P is denoted by r_1 and the corresponding centre of curvature by O_1. For the three-dimensional toroidal surface, there would be two principal radii of curvature (being the maximum and minimum values of curvature) at any given point P, and these occur in planes perpendicular to each other. The first principal radius of curvature of the toroidal surface at point P is the radius of curvature r_1 ($=PO_1$) of the generator ellipse at that point, while the second principal radius of curvature at point P, denoted by r_2, is equal to the distance PO_2, where O_2 is the point at which the surface normal at P intersects the axis of revolution $Y-Y$ of the torus.

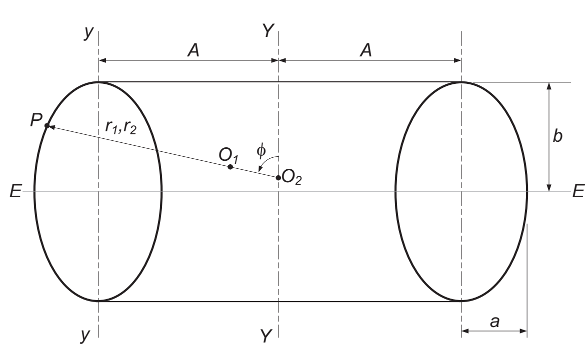

Fig. 1. Geometrical parameters of an elliptic toroidal shell.

Point P itself may be defined by an angular coordinate ϕ, which is the angle measured from the upward direction of the axis of revolution of the torus to the surface normal at point P. The range $0 \le \phi \le 2\pi$ covers all points on the toroidal surface, with $0 \le \phi \le \pi$ describing points in the outer region of the torus, and $\pi \le \phi \le 2\pi$ describing points in the inner region of the torus; the coordinates $\phi = \pi/2$ and $\phi = 3\pi/2$ define points on the equatorial plane, which of course correspond to the extrados and intrados of the torus with respect to the axis $Y-Y$.

For the outer region of the torus ($0 \le \phi \le \pi$), the principal radii of curvature are given by [3]

$$r_1 = \frac{a^2 b^2}{\left(a^2 \sin^2\phi + b^2 \cos^2\phi\right)^{3/2}} \text{ (positive)} \tag{1a}$$

$$r_2 = \frac{A}{\sin\phi} + \frac{a^2}{\left(a^2 \sin^2\phi + b^2 \cos^2\phi\right)^{1/2}} \text{ (positive)} \tag{1b}$$

while for the inner region ($\pi \le \phi \le 2\pi$), these become

$$r_1 = \frac{-a^2 b^2}{\left(a^2 \sin^2\phi + b^2 \cos^2\phi\right)^{3/2}} \text{ (negative)} \tag{2a}$$

$$r_2 = \frac{A}{|\sin\phi|} - \frac{a^2}{\left(a^2 \sin^2\phi + b^2 \cos^2\phi\right)^{1/2}} \text{ (positive)} \tag{2b}$$

The values of r_1 and r_2 at the extrados of the torus ($\phi = \pi/2$) and the intrados ($\phi = 3\pi/2$), which correspond to the outer and inner sides of the elliptical section, will be required in due course. Evaluating these, we obtain

$$r_1 = \frac{b^2}{a} \tag{3a}$$

$$r_2 = A + a \tag{3b}$$

at the extrados, and

$$r_1 = -\frac{b^2}{a} \tag{4a}$$

$$r_2 = A - a \tag{4b}$$

at the intrados.

3. Governing equation

Fig. 2 shows a bending element of an axisymmetrically-loaded shell of revolution in the $\{\phi, \theta\}$ coordinate system. Here, the meridional angle ϕ identifies the position of a point along a given