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An approximate analytical procedure for natural vibration analysis of free rectangular plates

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ABSTRACT

Natural vibrations of free rectangular plates are usually analysed by numerical methods since it is not possible to obtain the closed form analytical solution. In this paper a simple analytical procedure for estimation of natural frequencies of free thin rectangular plates, based on the Rayleigh's quotient, is presented. First, natural modes are assumed in the usual form as products of beam natural modes in longitudinal and transverse direction, satisfying the grillage boundary conditions. Based on a detailed FEM analysis some additional natural modes are recognized, which are defined as sum and difference of the cross products of beam modes. Their frequency spectra are very close and identical in some special cases manifesting in such a way double frequency phenomenon. These three families of natural mode shapes form a complete natural frequency spectrum of a free rectangular plate as a novelty. The reliable approximation of natural modes enables application of the Rayleigh's quotient for estimation of higher natural frequencies. Application of the developed procedure is illustrated in the case of a free thin square and rectangular plate. The obtained results are compared with those determined by FEM and also with more rigorous ones from the relevant literature based on the Rayleigh–Ritz method. The achieved accuracy is acceptable from the engineering point of view, and the procedure can be applied to improve the hydroelastic analysis of VLFS.

1. Introduction

The rectangular plate is a structural element used in many engineering structures. Thin and thick plates are distinguished, and are analysed by the Kirchhoff and Mindlin theory [1] and [2], respectively. For instance, thin plate is structural element of ship deck and bulkhead grillages, where one field between cross girders can be considered as an individual plate [3]. Recently very large floating structures (VLFS), i.e. floating airports, artificial islands etc., are treated as a large thin plate [4].

Vibration of thin rectangular plate is a classical problem analysed in a large number of papers and is already solved [5], while vibrations of thick plates are still being investigated [6] and [7]. In both cases the analytical solution is achieved for rectangular plate, which is simply supported at least at two opposite edges. For all other combinations of the boundary conditions (simply supported, clamped and free), numerical methods are used, as for instance the Rayleigh–Ritz method, or more often FEM due to its simplicity.

In 1973, Leissa [8] presented analytical solution for free vibrations of rectangular plate simply supported at two opposite edges. Also, problem of mixed boundary conditions (simply supported (S), clamped (C) and free (F)) is analysed by the Rayleigh–Ritz method assuming plate deflection as products of beam natural modes. Clamped and simply supported plate boundary conditions are exactly satisfied, while free edge conditions are only approximated, reducing in such a way the accuracy of the results. It is concluded that additional symmetry of the square plate increases confusion when identifying natural modes. Certain vibration modes have not been discovered in the relevant literature.

Mizusawa [9] analysed natural response of rectangular plates with free edges by the Rayleigh–Ritz method with B-spline functions, and investigated the effects of Poisson's ratio on natural frequencies for free-edge square plates. Differential quadrature element method is applied to vibration analyses of plates with free boundary conditions by Malik and Bert [10], while Wang et al. [11] utilized the similar approach for both static and dynamic consideration of the above problem. In order to overcome the difficulty of implementing the free boundary conditions in the discrete singular convolution (DSC) Wei and his collaborators, [12] and [13], have developed the method of matched interface and boundary (MIB), capable to analyse first several natural frequencies. Also, the discrete DSC method is applied by Wang and Xu [14] and comparisons with the above solutions are provided. Furthermore, in the context of applicability of related

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mathematical models to the analysis of dynamic response of very large floating structures (VLFS), Wang et al. [15] highlighted some problems in obtaining accurate modal stress-resultant distributions in freely vibrating plates analysed by conventional methods. Namely, they showed that if one adopts the classical thin plate theory and the Galerkin's method with commonly used modal functions consisting of the products of free-free beam modes, the natural boundary conditions are not satisfied at the free edges. Moreover, they indicated persistence of the mentioned problem within adoption of the refined Mindlin plate theory and use of NASTRAN software [16] (utilizing finite element method) or the Ritz method. Also, Wang et al. [15] demonstrate that a modified version of the Ritz method, which involves penalty functional for enforcement of the natural boundary conditions does not solve the problem when the plate is relatively thin, due to so called artificial stiffening of the plate. In order to overcome the above problem, Wang et al. [17] later proposed a mesh-free least squares-based finite difference method (LSFD) for evaluating vibration solutions of completely free plates, adopting not classical, but Mindlin plate theory.

Generally speaking, in the Rayleigh–Ritz method, natural modes are presented by polynomials with large number of terms resulting in a rather time consuming procedure. More effective is to assume physical natural modes in a series of mathematical modes. For this purpose, products of beam natural modes in longitudinal and transverse direction, satisfying the grillage boundary conditions, are usually used. In this case one mathematical mode is dominant. However, FEM vibration analysis of a free square plate shows that there are some modes of extraordinary shapes, which cannot be approximated by product of beam modes successfully, since none of the mathematical modes is dominant.

In order to overcome the above problem an analytical investigation of natural vibrations of a free thin rectangular plate is undertaken. Two additional families of extraordinary modes shapes are identified and described by sum and difference of beam natural modes, respectively. The Rayleigh's quotient is used not only for the first but also for the higher modes [18]. In such a way complete and denser spectrum of natural frequencies is obtained.

2. General solution of differential equation

A rectangular plate specified in the Cartesian coordinate system is considered, Fig. 1. Differential equation of natural vibrations is obtained by employing constitutive equilibrium equations of

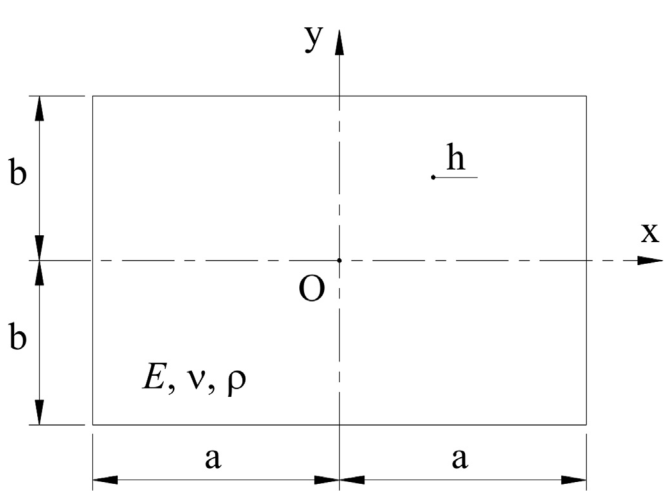

Fig. 1. Particulars of a thin rectangular plate.

sectional forces and moments, and Hooke's law for their dependence on change of curvature [5]

$$D\,\Delta\Delta w + \bar{m}\frac{\partial^2 w}{\partial t^2} = 0\,, \tag{1}$$

where $w = w\,(x,\ y,\ t)$ is deflection function, $D = Eh^3/[12(1-\nu^2)]$ is flexural rigidity, $\bar{m} = \rho\,h$ is mass per unit plate area, and $\Delta(\ .\ ) = \partial^2(\ .\ )/\partial x^2 + \partial^2(\ .\ )/\partial y^2$ is the Laplace differential operator. Natural vibrations are harmonic, i.e. $w = W e^{i\omega t}$ and Eq. (1) is reduced to the amplitude differential equation

$$\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} - \omega^2 \frac{\bar{m}}{D} W = 0\,, \tag{2}$$

where ω is natural frequency.

An ordinary differential equation has unique general solution and integration constants are determined later on by satisfying boundary conditions. On the contrary, partial differential equation can have a few general solutions and it is necessary to find such one which will satisfy given boundary conditions a priori. In the plate theory solution of Eq. (2) is ordinary assumed in the form of separated variables.

$$W(x,\ y) = X(x)Y(y)\,. \tag{3}$$

By substituting Eq. (3) into Eq. (2), yields

$$X''''Y + 2X''Y'' + X\,Y'''' - r^2 X\,Y = 0\,, \tag{4}$$

where $r^2 = \omega^2 \bar{m}/D$.

Solution for the separated functions is assumed in the exponential form

$$X = e^{\mu x}\ Y = e^{\lambda y}. \tag{5}$$

Their substitution into Eq. (4) leads to the characteristic equations

$$\left(\mu^2 + \lambda^2\right)^2 = r^2\,, \tag{6}$$

with two solutions

$$\mu^2 + \lambda^2 = \pm r. \tag{7}$$

Furthermore, each of Eq. (7) has two solutions, i.e. $\mu_{1,2} = \pm\,\alpha$, $\lambda_{1,2} = \pm\,\beta$, and $\mu_{3,4} = \pm\ i\,\alpha$, $\lambda_{3,4} = \pm\ i\,\beta$. Finally, by substituting the obtained roots into (5) one can write

$$X = A_1\,\mathrm{sh}\,\alpha x + A_2\,\mathrm{ch}\,\alpha x + A_3\ \sin\ \alpha x + A_4\ \cos\ \alpha x,$$
$$Y = B_1\,\mathrm{sh}\,\beta y + B_2\,\mathrm{ch}\,\beta y + B_3\ \sin\ \beta y + B_4\ \cos\ \beta y. \tag{8}$$

The even and the odd constants in (8) are related to the symmetric and antisymmetric modes, respectively.

3. Natural vibrations of free plate based on solution of differential equation

3.1. Boundary conditions

Boundary conditions for a free plate are related to the bending moments, torsional moments and shear forces. Since there are more conditions than unknown constants in solution (8), torsional moments are incorporated into the shear forces [5]. Hence, expressions for bending moments and effective shear forces are the following: