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Effect of higher order constitutive terms on the elastic buckling of thin-walled rods

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ABSTRACT

In this paper we revisit an elastic constitutive equation proposed in two previous works and extend it in order to include all higher-order terms on the deformations. Our purpose is to assess the influence of these terms on the elastic buckling of thin-walled rods. The resulting material model was incorporated into a geometrically exact rod formulation and implemented into a nonlinear finite element code. By means of simple numerical examples we show that the higher order terms may play a significant role on the values of the buckling loads and on the post-buckling behavior of thin-walled beams and columns.

1. Introduction

Thin-walled rod structures undergoing large displacements and large rotations are very common in engineering practice. The development of geometrically exact kinematical models for such rod assemblages has consequently attracted much attention in recent decades, and various papers have been published on the subject. We mention here the seminal articles of Simo and Vu-Quoc in [1,2], and the many works that followed (e.g. [3–18]), to cite just a few.

Due to the geometrical characteristics of these types of rods, buckling and cross-sectional warping are important aspects that must be addressed in their modeling. However, not only the kinematical description of the rod must be able to account for these aspects, but also the constitutive equation must allow for a proper coupling of the strains in order to adequately capture the rod's behavior.

Linear elasticity is by far the most widely used constitutive assumption in the nonlinear analysis of rod structures. Accordingly, the stresses are related to the strains by means of a constant constitutive matrix (constant in the sense that it does not depend on the rod's strains). This matrix is a function only of the elasticity moduli E and G of the material and of the trivial geometrical properties of the cross-sections (area, first and second moments of inertia, product of inertia, torsion inertia and, if warping is considered, warping inertia). What is tricky when large displacements and large rotations are present is that, differently from the classical case of linear kinematics with small deformations, the stress–strain work-conjugate pair that is to be related to each other is not uniquely defined. Many conjugate pairs are indeed possible (provided frame-invariance is observed) and enforcing a linear relation between a given pair does not necessarily mean that the relation will render linear for another. This aspect, if not addressed properly, may lead to insufficient coupling of the strains and thus give rise to inaccurate solutions in certain types of problems. Such issue has been observed in the comprehensive works of [19–22]. Therein, although through different rod formulations, a simple idea has been proposed to overcome it: if some specific second-order strain terms are incorporated into the constitutive relation (or if an appropriate conjugate pair is selected), the desired coupling eventually shows up and prevents the misbehavior. The coupling terms came to be identified with the often called "Wagner terms", and their effects ended up being called "Wagner effects" in the context of thin-walled rods.

In [19], specifically, it has been noticed that by adopting the classical Kirchhoff–St.-Venant relation between second Piola–Kirchhoff stresses and linearized Green–Lagrange strains one may attain significantly inconsistent results in the modeling of torsion buckling problems (i.e., in the modeling of the torsion mode of buckling in compressed columns). The lack of coupling

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between torsion and compression strains that followed from this model was identified and the corresponding correcting (quadratic) terms were proposed. In fact, quadratic terms were proposed for *all* strains of the formulation. This was performed within a rod model that involved six degrees-of-freedom (three displacements and three rotations), i.e., within a kinematical description where cross-sectional warping was not explicitly taken into account, and proved to partially fix the problem. The resulting constitutive equation was called therein the "quadratic Saint-Venant material".

With the aim to study the lateral buckling of beams, in [23] the above ideas were extended to a seven degrees-of-freedom rod model (the seventh dof being the warping degree-of-freedom). Due to the complexity of dealing with the warping-related strains at the constitutive level, however, the quadratic terms were considered only for (but all of) the non-warping strains. Similar ideas were proposed by [20,22] (although with fewer, judiciously chosen, quadratic terms). This lead in [23] to the so-called "modified quadratic St.-Venant material", but proved to work only in certain kinds of lateral buckling problems. It was observed that the lateral post-buckling behavior was not always possible to be traced, due to a lack of convergence of the numerical scheme. This aspect was attributed to the fact that the Kirchhoff–St.-Venant material law is not a polyconvex material (see e.g. [24] for an discussion on polyconvexity), and therefore the solution to boundary value problems involving this law may not always be attained – especially if the strains enter the moderate to large regime, what can be the case in developed post-buckling stages.

This drawback led the author in [23] to adopt the polyconvex material model of Simo–Ciarlet ([24,25]) as the starting point, in place of the Kirchhoff–St.-Venant's. Analogous derivations as described above were conducted, arriving at the so-called "modified quadratic Simo–Ciarlet material". The problem with the lack of convergence was then fixed, however the values observed for the lateral buckling loads were not always consistent. To be more specific, with the use of this constitutive equation the same inconsistent buckling loads as with the linearized material were obtained, both in problems of lateral buckling of beams and of torsion buckling of columns. A controversial observation was thus faced: on one hand, the modified quadratic Saint-Venant material gives correct values for the buckling loads but limited or zero response in the post-buckling regime; on the other hand, the modified quadratic Simo-Ciarlet material gives inconsistent values of the buckling loads but full post-buckling response. In a further attempt to fix this issue, in [26] the authors incorporated the second-order warping strain terms into both modified quadratic constitutive equations, rendering "fully quadratic" material models. Yet, this did not correct the inconsistencies.

In this context, the purpose of this work is to present an "exact" elastic constitutive equation ("exact" in the sense that it contains all higher-order terms on the deformations) for the analysis of thin-walled rods with warping degrees of freedom. Differently from [19,23,26], we start from the three-dimensional continuum mechanics form of the Kirchhoff–St.-Venant material law, relating the second Piola–Kirchhoff stress tensor to the Green–Lagrange strain tensor. Such a 3D approach (and for the same work-conjugate pair) is also found in the works of [10,17,20]. Here, however, in contrast to these latter and in fact to all other works we have found in the literature, all higher-order terms on the Green–Lagrange strains are preserved in the derived expressions of the stresses. These stresses are then transformed into nominal stresses (i.e. into first Piola–Kirchhoff stresses) and integrated over the rod's cross-section, rendering consistent cross-sectional stress-resultants. The corresponding elastic tangent moduli (i.e. the derivatives of these stress resultants with respect to the cross-sectional strains) are obtained in an exact manner.

Since we work with all terms on the deformations, the integrals that define the stress-resultants become very difficult (if not impossible) to be evaluated analytically (this would generate several cross-sectional higher-order geometrical properties with intricate integrals, especially due to the consideration of warping). We overcome this by performing numerical integration. However, the explicit expression of the cross-sectional warping function turns to be necessary, and this places a drawback within our model. We propose a simple, yet consistent, warping function that is well-suited for thin-walled sections. The resulting constitutive equation was incorporated into the geometrically exact 7-dof rod model of [6] (which is a generalization of the 6-dof formulation of [5]) and then implemented into a nonlinear finite element code. Buckling problems of thin-walled beams and columns are analyzed to assess the extent of the formulation.

The paper is organized as follows: in Section 2 we present a brief description of the geometrically exact rod kinematical model adopted herein (this was found necessary in order to show many of the expressions that we need to build our "exact" constitutive equation); in Section 3 we derive our constitutive formulation and incorporate it into the rod model of Section 2; in Section 4 we introduce our warping function for thin-walled cross-sections; in Section 5 we show a few (but illustrative) numerical examples; and in Section 6 we derive our conclusions.

Throughout the text, italic Greek or Latin lowercase letters $(a, b, \ldots, \alpha, \beta, \ldots)$ denote scalar quantities, bold italic Greek or Latin lowercase letters $(\boldsymbol{a}, \ \boldsymbol{b}, \ \ldots, \boldsymbol{\alpha}, \boldsymbol{\beta}, \ldots)$ denote vectors and bold italic Greek or Latin capital letters $(\boldsymbol{A}, \ \boldsymbol{B}, \ \ldots)$ denote second-order tensors in a three-dimensional Euclidean space. Summation convention over repeated indices is adopted, with Greek indices ranging from 1 to 2 and Latin indices from 1 to 3.

2. Brief description of the rod kinematics

This section outlines the rod kinematical model that we adopt as the basis for our developments. It has origins in the purely theoretical work of [6] and was first implemented in [23]. It consists of a geometrically exact formulation in which (i) shear deformation due to bending and (ii) cross-section warping due to combined bending/non-uniform torsion are explicitly taken into account.

A straight reference configuration is assumed for the rod axis at the outset. A local orthonormal system $\{\boldsymbol{e}_1^r, \boldsymbol{e}_2^r, \boldsymbol{e}_3^r\}$ with corresponding coordinates $\{x_1, x_2, x_3\}$ is defined in this configuration, with vectors $\boldsymbol{e}_\alpha^r$ $(\alpha = 1, 2)$ placed on the rod cross-section and $\boldsymbol{e}_3^r$ placed along the rod axis (see Fig. 1). Points in this configuration are described by the vector field

$$\boldsymbol{\xi} = \boldsymbol{\zeta} + \boldsymbol{a}^r, \tag{1}$$

where $\boldsymbol{\zeta} = x_3 \boldsymbol{e}_3^r$ describes the position of points at the rod axis and $\boldsymbol{a}^r = x_\alpha \boldsymbol{e}_\alpha^r$ defines the position of points at the cross-section relative to the rod axis. Notice that $x_3 \in L = [0, \ell]$ is the axis coordinate, with ℓ being the rod reference length.

In the current configuration another local orthonormal system $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is defined, as depicted in Fig. 1. The rod deformation is then described by a vector field $\boldsymbol{x}$ such that the position of the material points is expressed by

$$\boldsymbol{x} = \boldsymbol{z} + \boldsymbol{a} + p\psi \boldsymbol{e}_3, \tag{2}$$

where $\boldsymbol{z} = \hat{\boldsymbol{z}}(x_3)$ describes the position of points at the deformed axis, $\boldsymbol{a} = \hat{\boldsymbol{a}}(x_\alpha, x_3)$ defines the position of points at the deformed cross-section *in the projection of its plane,* $\psi = \hat{\psi}(x_\alpha)$ is a function defining the warping of the cross-section with respect to its shear center (the so-called warping function) and $p = \hat{p}(x_3)$ is a scalar parameter that gives ψ its amplitude.