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Energy equations for elastic flexural–torsional buckling analysis of plane structures

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ABSTRACT

Lateral–torsional buckling is a critical mode of failure of metal structures. When the values of the loadings on a member of a structure reach a limiting state, the member will experience out-of-plane bending and twisting. This type of failure occurs suddenly in members with a much greater in-plane bending stiffness than torsional or lateral bending stiffness. Slender members of a structural system may buckle laterally and twist before their in-plane capabilities can be reached. Energy equations are derived by considering the total potential energy of a beam-column element. The second variation of the total potential energy equal to zero indicates the transition from a stable state to an unstable state, which is the critical condition for buckling. Several energy equations are derived analytically by calculating the second variation of the total potential energy of a double symmetric thin wall beam-column element. In this article, in-plane deformations of the beam-column element are disregarded. Then energy equations are derived expressing in dimensional and non-dimensional forms. These energy equations will be implemented in a future article to derive elastic and geometric stiffness matrices for the beam-column element and calculate the lateral–torsional buckling of plane structures. Examples are provided to show the accuracy of the equations and applications.

1. Introduction

In steel structures, all members in a frame are essentially beam-columns. Beam-columns are typically loaded in the plane of the weak axis so that bending occurs about the strong axis, such as in the case of the commonly used wide flange sections. Primary bending moments and in-plane deflections will be produced by the end moments and transverse loadings of the beam-column, while the axial force will produce secondary moments and additional in-plane deflections.

When the values of the loadings on the beam-column reach a limiting state, the member will experience out-of-plane bending and twisting. This type of failure occurs suddenly in members with a much greater in-plane bending stiffness than torsional or lateral bending stiffness. The limit state of the applied loads of an elastic slender beam of perfect geometry is called *the elastic lateral–torsional buckling load*. In a beam-column or plane frame structure, the buckling load may be referred to as *the elastic flexural–torsional buckling load.*

The flexural–torsional buckling load of a member is influenced by several factors including: (1) the cross section of the member, (2) the unbraced length of the member, (3) the support conditions, (4) the type and position of the applied loads, and (5) the location of the applied loads with respect to the centroidal axis of the cross section. The goal of a stability analysis is to consider these factors to determine the flexural–torsional buckling loads of a structure. If the flexural–torsional buckling loads of a structure are known, it may be necessary to design the member against flexural–torsional buckling by changing the member size or adding some bracings.

The energy method can be used to analyze and calculate the flexural–torsional buckling loads of simple structures such as a beam-column element. However, this method will involve excessive computations when applied to larger structures. Computational methodology based on computer technology may be needed in order to analyze more complicated flexural–torsional buckling problems.

2. Research plan

In this work, the finite element method is applied in conjunction with the energy equations to analyze flexural–torsional buckling of plane structures and provide acceptable results for buckling load calculations. There are several approaches to derive stiffness matrices and element equations when applying

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Nomenclature

A	area of member
a	distributed load height
$\bar{a}$	non-dimensional distributed load height
E	modulus of elasticity
e	concentrated load height
$\bar{e}$	non-dimensional concentrated load height
F	axial load
$\bar{F}$	non-dimensional axial load
G	shear modulus
h	depth of the member
I_x	moment of inertia about the x axis
I_y	moment of inertia about the y axis
I_ω	warping moment of inertia
J	torsional constant
K	beam parameter
k_z	torsional curvature of the deformed element
L	member length
$M_x(z)$	bending moment
M_1	moment at node 1
M_2	moment at node 2
$\bar{M}_1$	non-dimensional moment at node 1
P	concentrated load
$\bar{P}$	non-dimensional concentrated load
q	distributed load
$\bar{q}$	non-dimensional distributed load
$[T_R]$	rotation transformation matrix
t_p	perpendicular distance to P from the mid-thickness surface
U	strain energy
U_e	strain energy for each finite element
u	out-of-plane lateral displacement
u_p	out-of-plane lateral displacement of point P_o
u_1, u_3	out-of-plane lateral displacements at nodes 1 and 2
u_2, u_4	out-of-plane rotation at nodes 1 and 2
u'	out-of-plane rotation
$\bar{u}$	non-dimensional out-of-plane lateral displacement
V_1	shear at node 1
V_2	shear at node 2
$\bar{V}_1$	non-dimensional shear at node 1
v	in-plane bending displacement
v_M	displacement through which the applied moment acts
v_P	displacement through which the concentrated load acts
v_p	in-plane bending displacement of point P_o
v_q	displacement through which the distributed load acts
v_1, v_3	in-plane displacements at nodes 1 and 2
v_2, v_4	in-plane rotation at nodes 1 and 2
v'	in-plane rotation
w	axial displacement
w_F	longitudinal displacement through which the axial load acts
w_p	longitudinal displacement of point P_o
z_P	concentrated load location from left support
$\bar{z}$	non-dimensional member distance
$\bar{z}_p$	non-dimensional distance to concentrated load
ε_p	longitudinal strain of point P_o
$\{\varepsilon_u\}$, $\{\varepsilon_v\}$	generalized strain vectors
ϕ	out-of-plane twisting rotation
ϕ_1, ϕ_3	out-of-plane twisting rotation at nodes 1 and 2
ϕ_2, ϕ_4	out-of-plane torsional curvature at nodes 1 and 2
ϕ'	out-of-plane torsional curvature
γ_p	shear strain of point P_o
Π	total potential energy
$\bar{\Pi}$	non-dimensional total potential energy
σ_p	longitudinal stress of point P_o
τ_p	shear stress of point P_o
ω	warping function
Ω	potential energy of the loads
θ	rotation of the member cross section

finite element method. Although all of these methods provide the same result, the stiffness matrices and element equations for complex elements can be derived much easier when applying a work or energy method.

In order to achieve the objective of this research in a logical fashion, the project is divided to three phases: (1) derivation of the energy equations for finite element applications; (2) derivation of the element elastic and geometric stiffness matrices, transformation from local to global coordinates and assembly process to obtain a generalized eigenvalue problem that must be solved to provide flexural–torsional buckling load for beam-columns and plane frames; and (3) software development using the finite element method and object-oriented technology.

Phase 1 of the research consists of two parts: (a) derivation of energy equations for lateral, flexural–torsional buckling load of plane structures when in-plane deformations are disregarded, (b) derivation of energy equations for elastic lateral, flexural–torsional buckling load of plane structures when in-plane deformations are considered. Part (a) is the subject of this article. Part (b) is presented in Torkamani and Roberts [1]. Phases 2 and 3 will be covered in future articles.

3. Variational methods

Energy equations are derived by considering the total potential energy of a beam-column element. The total potential energy of a structure, Π, is the sum of the strain energy, U, and the potential energy of the external loads, Ω, given by

$$\Pi = U + \Omega \tag{1}$$

Langhaar [2] and Brush et al. [3] showed the total potential energy increment may be expressed in the form:

$$\Delta\Pi = \delta\Pi + \frac{1}{2!}\delta^2\Pi + \frac{1}{3!}\delta^3\Pi + \cdots \tag{2}$$

where terms on the right-hand side are linear, quadratic, cubic, etc., respectively, in the infinitesimally small variational displacements. The theorem of stationary total potential energy states that an equilibrium position is one of stationary total potential energy which is expressed as

$$\delta\Pi = 0 \tag{3}$$

The theorem of minimum total potential energy states that the stationary value of Π (for which $\delta\Pi = 0$) of an equilibrium position is minimum when the position is stable. Therefore, the equilibrium position is stable when

$$\tfrac{1}{2}\delta^2\Pi > 0 \tag{4}$$

Consequently, the equilibrium position is unstable when

$$\tfrac{1}{2}\delta^2\Pi < 0 \tag{5}$$

The second variation of the total potential energy equal to zero indicates the transition from a stable state to an unstable state,