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The orthogonal meshless finite volume method for solving Euler–Bernoulli beam and thin plate problems

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ABSTRACT

In this paper a new method untitled "orthogonal meshless finite volume method" (OMFVM) is developed for solving elastostatic problems in Euler–Bernoulli beam and thin plate. In this method, the weak formulation of a conservation law is discretized by restricting it to a discrete set of test functions. In contrast to the usual finite volume approach, the test functions are not taken as characteristic functions of the control volumes in a spatial grid, but are chosen from a Heaviside step function. The present approach eliminates the expensive process of directly differentiating the OMLS interpolations in the entire domain. This method was evaluated by applying the formulation to a variety of patch test and thin beam problems. The formulation successfully reproduced exact solutions. Numerical examples demonstrate the advantages of the present methods: (i) lower-order polynomial basis can be used in the OMLS interpolations; (ii) smaller support sizes can be used in the OMFVM approach; and (iii) higher accuracies and computational efficiencies are obtained.

1. Introduction

The orthogonal meshless finite volume method (OMFVM) is a new meshless method for the discretization of governing differential equations. The motivation for developing this new method is to unify advantages of meshless methods and finite volume methods (FVM) in one scheme. The basic idea in the OMFVM is to incorporate elements of the FVM into an orthogonal moving least-square (OMLS) method [1,2].

Meshless methods are very flexible because they are free of using mesh. The need for meshless methods typically arises if problems with time dependent or very complicated geometries are under consideration because the handling of mesh discretization becomes technically complicated or very time consuming. Fluid flows with structural interaction or fast moving boundaries like an inflating air-bag are of that kind for instance.

Advantages of meshless methods are to overcome some of the disadvantages of mesh-based methods such as discontinuous secondary variables across inter-element boundaries and the need for remeshing in large deformation problems [3–8]. Extensive research on meshless methods, in particular, the meshless local Petrov–Galerkin (MLPG) method recently exists in literatures. There is analysis of thin beam problems using a Galerkin implementation of the MLPG method [8–13]; a generalized moving least squares (GMLS) approximation is used to construct the trial functions, and the test functions are chosen from the same space. Refs. [9 and 10] showed good performance of the MLPG method for potential and elasticity problems and a good performance for beam problems. However, these methods need a large number of calculations to compute the first and second order derivatives of the moving least squares (MLS) trial functions that are required in the weak form and special procedures were needed to integrate the weak form accurately.

The purpose of this paper is to develop and use of the OMFVM for thin beam and plate problems. The method is evaluated by applying the formulation to patch test and mixed boundary value problems and problems with complex loading conditions.

The outline of the paper is as follows. First, the OMLS interpolation scheme is described and then the FV form of the governing differential equation is derived in a general sense, and a system of algebraic equations is developed from this FV form. Next, the OMLS method is used to discretize these formulations and to obtain the OMFVM form of the governing differential equation. Finally, the performance of the OMFVM is investigated by implying to some patch test and examples.

2. Meshless interpolation

In general, meshless methods use a local interpolation, or an approximation, to represent the trial function, using the values (or the fictitious values) of the unknown variable at some

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randomly located nodes in the local vicinity. The moving least-square method is generally considered to be one of the best schemes to interpolate data with a reasonable accuracy. Basically the MLS interpolation does not pass through the nodal data. Consider a domain in question with control points for boundaries (i.e. nodes on boundaries) and some scattered nodes inside, where every node has its undetermined nodal coefficient (fictitious nodal value) and an influence radius (radius for local weight function). Now for the distribution of trial function at any point x and its neighborhood Ω_s located in the problem domain Ω, $u^h(x)$ may be defined by

$$u^h(x) = p^T(x)\mathbf{a}(x) \quad \forall x \in \Omega_s \tag{1}$$

The coefficient vector $\mathbf{a}(x)$ is determined by minimizing a weighted discrete L_2 norm, which can be defined as

$$J(x) = \sum_{I=1}^{N} w_I(x)\left[\mathbf{p}(x_I)\mathbf{a}(x) - \hat{u}^I\right]^2 \tag{2}$$

where $w_I(x)$, is a weight function associated with the node I, with $w_I(x) > 0$ for all x in the support of $w_I(x)$, x_I denotes the value of x at node I, N is the number of nodes in Ω_s for which the weight functions $w_I(x) > 0$. Here it should be noted that $\hat{u}_I$, $I = 1, 2, \ldots, N$, in Eq. (4), are the fictitious nodal values (undetermined nodal coefficients), and not the exact nodal values of the unknown trial function $u^h(\mathbf{x})$, in general.

Solving for $\mathbf{a}(x)$ by minimizing J in Eq. (2), and substituting it into Eq. (1), give a relation which may be written in the form of an interpolation function similar to that used in the FEM, as

$$u^h(x) = \sum_{I=1}^{N} \phi^I(x)\hat{u}^I \quad u^h(x_I) \equiv u^I \neq \hat{u}^I, \quad x \in \Omega_s \tag{3}$$

where

$$\phi^I(x) = \sum_{j=1}^{m} p_j(x)\left[\mathbf{A}^{-1}(x)\mathbf{B}(x)\right]_{jI} \tag{4}$$

with the matrix $\mathbf{A}(x)$ and $\mathbf{B}(x)$ being defined by

$$\mathbf{A}(x) = \sum_{I=1}^{N} w_I(x)p(x_I)p^T(x_I) \tag{5}$$

$$\mathbf{B}(x) = [w_1(x)p(x_1), w_2(x)p(x_2), \ldots, w_N(x)p(x_N)]. \tag{6}$$

The nodal shape function is complete up to the order of the basis. The smoothness of the nodal shape function $\Phi^I(x)$ is determined by that of the basis and of the weight function. The MLS approximation is well defined only when the matrix $\mathbf{A}$ in Eq. (5) is non-singular. The shape function may be found as

$$u(x) = p^T(x)\mathbf{A}^{-1}(x)\mathbf{B}(x)u = \Phi^T(x)u \quad \forall x \in \Omega_s \tag{7}$$

If Eq. (6) has the properties of linearity, symmetry and positive-definiteness it is a weighted inner product space. Using orthogonality $\mathbf{A}(x)$ can be changed to a diagonal matrix which means its inverse can be obtained without having singularity problem. Therefore matrix $\mathbf{A}(x)$ can be a weighted orthogonal as

$$\mathbf{A}_{jk}(x) = P_j^T W P_k = \sum_{i=1}^{n} w_i(x)p_k(x_i)p_j^T(x_i) = \begin{cases} 0 & j \neq k \\ \overline{A}_j(x) & j = k \end{cases} \quad j,k = 1,2,\ldots,m \tag{8}$$

In other word, matrix $\mathbf{A}(x)$ will be in the following form:

$$\mathbf{A}(x) = \begin{bmatrix} \overline{A}_1 & 0 & \ldots & 0 \\ 0 & \overline{A}_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & \overline{A}_m \end{bmatrix} \tag{9}$$

it is clear $\mathbf{A}(x)$ is a diagonal matrix so its inverse can be obtained as

$$\mathbf{A}^{-1}(x) = \begin{bmatrix} \frac{1}{\overline{A}_1} & 0 & \ldots & 0 \\ 0 & \frac{1}{\overline{A}_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & \frac{1}{\overline{A}_m} \end{bmatrix} \tag{10}$$

In order to obtain a weighted orthogonal $\mathbf{A}(x)$ the following monomial basis p(x) can be considered

$$p_1 = 1 \tag{11a}$$

$$p_2 = r - \frac{(rp_1, p_1)}{(p_1, p_1)} \tag{11b}$$

$$p_i = \left(r - \frac{(rp_{i-1}, p_{i-1})}{(p_{i-1}, p_{i-1})}\right)p_{i-1} - \frac{(p_{i-1}, p_{i-1})}{(p_{i-2}, p_{i-2})}p_{i-2} \quad i = 3,4,5,\ldots \tag{11c}$$

where $r = \sum_{i=1}^{3} x_i$.

The choice of the weight function is more or less arbitrary as long as the weight function is positive and continuous. The following weight function is considered in the present work:

$$w_I(x) = \begin{cases} 1 - 6\left(\frac{d_I}{r_I}\right)^2 + 8\left(\frac{d_I}{r_I}\right)^3 - 3\left(\frac{d_I}{r_I}\right)^4 & 0 \leq d_I \leq r_I = \rho_I h_I \\ 0 & d_I > r_I = \rho_I h_I \end{cases} \tag{12}$$

where $d_I = |x - x_I|$ is the distance from node x_I to point x, h_I is the nodal distance, and ρ_I is the scaling parameter for the size of the sub-domain Ω^I_{tr}.

3. Local weak form and OMFVM for Euler–Bernoulli beam

Consider the governing equation of an Euler–Bernoulli beam [14]

$$EI\,u'''' = f \quad \text{in global domain } \Omega \tag{13}$$

where u is transverse displacement, EI denotes the bending stiffness, and f is distributed load over the beam. The boundary conditions are given at the global boundary, Γ, as

$$u(x) = \overline{u}(x) \text{ on } \Gamma_u, \text{ and } \frac{\partial u(x)}{\partial x} = \overline{\theta}(x) \text{ on } \Gamma_\theta \tag{14a}$$

$$M = \overline{M} \text{ on } \Gamma_M, \text{ and } V = \overline{V} \text{ on } \Gamma_V \tag{14b}$$

where M and V denote the moment and the shear force, respectively. Γ_u, Γ_θ, Γ_M and Γ_V denote the boundary regions where displacement, slope, moment, and shear force are specified, respectively. The moment and shear force are related to the displacement through the equations

$$M = EIu'' \text{ and } V = -EIu''' \tag{15}$$

Different from the other meshless methods, such as the element free Galerkin method, which are based on the global weak formulation over the entire domain Ω, a local weak form over a local sub-domain Ω_s located entirely inside the global domain Ω will be used in this study. It is noted that the local sub-domain can be of an arbitrary shape containing a point x in question. Even though a particular approximation of the local weak form will give the same resulting discretized equations as from the Galerkin approximation of global weak form, the local weak form will provide the clear concept for a local non-element integration, which does not need any background integration cell over the entire domain. And, it will lead to a natural way to construct the global stiffness matrix, not through the integration over a global domain, but through the integration over a local sub-domain.