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A sector elliptic *p*-element applied to membrane vibrations

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ABSTRACT

A new sector *p*-element is derived and implemented in elliptic coordinates. The element is applied to the free vibration analysis of annular elliptic membranes. The internal shape functions are derived from the shifted Legendre orthogonal polynomials. The stiffness and mass matrices may be integrated exactly using symbolic computing. One-quarter of the annular elliptic membrane is modeled as one element. The solution of the whole membrane is obtained from the solution of one-quarter with appropriate boundary conditions along the symmetry lines. The accuracy of the solution is improved simply by increasing the polynomial degree. Values for the natural frequencies of annular elliptic membranes are obtained and compared with published results. Comparisons show good agreement. New highly accurate values for the natural frequencies of annular elliptic membranes with different aspect ratios and boundary conditions are presented. A case of a sector annular elliptic membrane is also shown.

1. Introduction

Membranes are important components of machines and acoustical instruments. Closed-form solutions for the natural frequencies of rectangular membranes may be expressed in Cartesian coordinates in terms of trigonometric functions and for circular membranes in polar coordinates in terms of Bessel functions [1]. Closed-form solutions for the natural frequencies of annular elliptic membranes may be expressed in elliptic coordinates in terms of Mathieu functions. However, a complete evaluation of these functions is rarely available and resort to approximate methods such as the finite element method is made. Buchanan and Peddieson [2] presented a sector elliptic finite element for the free vibration analysis of elliptic membranes. A nine-node Lagrangian element has been derived and implemented in elliptic coordinates. Results have been presented for solid and annular elliptic membranes. The annular elliptic membrane was defined as a conformal ellipse. Both free and fixed inner boundary conditions were analyzed.

The free vibration analysis of membranes of polygonal shape by means of the *p*-version of the finite element method also known as the hierarchical finite element method has been investigated by the author [3,4]. Rectangular and triangular *p*-elements were derived and implemented in Cartesian coordinates. Numerical results have shown that *p*-elements produce a higher accuracy than conventional finite elements with fewer degrees of freedom and this motivated the need to derive and implement a *p*-element in elliptic coordinates, and to apply it to the free vibration analysis of annular elliptic membranes.

This paper presents an analysis of the free vibration of annular elliptic membranes using a new polynomially enriched sector elliptic *p*-element. The element describes the geometry of the annular elliptic membrane exactly and is, therefore, suitable for this type of membrane. The shifted Legendre orthogonal polynomials are used to derive the internal shape functions. The integrals in the stiffness and mass matrices are calculated exactly with the aid of symbolic computing. In this method, one-quarter of the annular elliptic membrane is modeled as just one sector elliptic *p*-element. The solution of the whole membrane is obtained from the solution of one-quarter with appropriate boundary conditions along the symmetry lines. The accuracy of the solution is improved simply by increasing the polynomial degree while fixing the mesh. Results for the natural frequencies are obtained for annular elliptic membranes with different boundary conditions. Comparisons are made with values from Ref. [2]. New highly accurate values for the natural frequencies of annular elliptic membranes with different aspect ratios and boundary conditions are presented. A case of a sector annular elliptic membrane is also shown.

2. Formulation

2.1. The shape functions

They consist of vertex and internal shape functions. The vertex shape functions are used to describe the degrees of freedom at the

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Nomenclature

x, y	Cartesian coordinates
β, α	elliptic coordinates
ξ, η	non-dimensional coordinates
γ	one-half the focal length
a	major semi-axis
b	minor semi-axis
c	major semi-axis for the opening
d	minor semi-axis for the opening
t	time
w	out-of-plane displacement
p	polynomial degree
$P_p{}^*$	shifted Legendre orthogonal polynomial of degree p
ρ	surface density
S	tension (per unit length)
U	strain energy
T	kinetic energy
$\mathbf{K}$	stiffness matrix
$\mathbf{M}$	mass matrix
$\mathbf{Q}$	displacement vector
ω	natural frequency
Ω	$=\omega b\sqrt{\rho/S}$, non-dimensional frequency parameter

vertices. The internal shape functions are used to provide additional freedom to the edges and the interior of the element.

The vertex shape functions are

$$g_1(\zeta) = 1-\zeta, \tag{1}$$

$$g_2(\zeta) = \zeta. \tag{2}$$

Various sets of polynomials can be selected for the internal shape functions provided the set is complete. The polynomials used here are forms of the shifted Legendre orthogonal polynomials [5].

The internal shape functions $g_{p+1}(\zeta)$ are defined as

$$g_{p+1}(\zeta) = \sqrt{2p-1}\int_0^{\zeta} P^*_{p-1}(\tau)\,\mathrm{d}\tau. \tag{3}$$

The functions $g_{p+1}(\zeta)$ have the following orthogonality property:

$$\int_0^1 \frac{\mathrm{d}g_i(\zeta)}{\mathrm{d}\zeta}\frac{\mathrm{d}g_j(\zeta)}{\mathrm{d}\zeta}\,\mathrm{d}\zeta = \begin{cases} 1 & \text{if} \quad j=i, \\ 0 & \text{if} \quad j\neq i. \end{cases} \tag{4}$$

The first six internal shape functions $g_{p+1}(\zeta)$ ($p = 2, 3, \ldots, 7$) are quoted explicitly in Table 1 and plotted in Fig. 1.

2.2. *The sector elliptic p-element*

Elliptic coordinates are defined as traces of confocal ellipses and hyperbolas in the x–y plane, as shown in Fig. 2. Transformations between Cartesian and elliptic coordinates are

$$x = \gamma \cosh \beta \cos \alpha, \tag{5}$$

$$y = \gamma \sinh \beta \sin \alpha, \tag{6}$$

where

$$\gamma = \sqrt{a^2 - b^2} \quad (a > b). \tag{7}$$

The geometric configuration of the sector elliptic p-element is shown in Fig. 3. The element is defined as a sector of an annular confocal ellipse. The degree of freedom per vertex and per edge is the out-of-plane displacement. The element is defined in terms of elliptic coordinates β and α. Also, non-dimensional coordinates ξ and η are used with the following relationships between the two:

$$\xi = \frac{\beta-\beta_0}{\beta_1-\beta_0}, \tag{8}$$

$$\eta = \frac{\alpha}{\alpha_1}. \tag{9}$$

The constants β_0 and β_1 are given by

$$\beta_0 = \frac{1}{2}\ln\left(\frac{c+d}{c-d}\right), \tag{10}$$

$$\beta_1 = \frac{1}{2}\ln\left(\frac{a+b}{a-b}\right), \tag{11}$$

where

$$c = \sqrt{\gamma^2 + d^2}. \tag{12}$$

Using the p-version of the finite element method, the out-of-plane displacement within the element will be approximated by

$$w = \sum_{k=1}^{p+1}\sum_{l=1}^{p+1} q_n g_k(\xi) g_l(\eta), \tag{13}$$

Table 1
The first six internal shape functions $g_{p+1}(\zeta)$ ($p = 2, 3, \ldots, 7$)

p	$g_{p+1}(\zeta)$
2	$\sqrt{3}(\zeta^2-\zeta)$
3	$\sqrt{5}(2\zeta^3-3\zeta^2+\zeta)$
4	$\sqrt{7}(5\zeta^4-10\zeta^3+6\zeta^2-\zeta)$
5	$3(14\zeta^5-35\zeta^4+30\zeta^3-10\zeta^2+\zeta)$
6	$\sqrt{11}(42\zeta^6-126\zeta^5+140\zeta^4-70\zeta^3+15\zeta^2-\zeta)$
7	$\sqrt{13}(132\zeta^7-462\zeta^6+630\zeta^5-420\zeta^4+140\zeta^3-21\zeta^2+\zeta)$

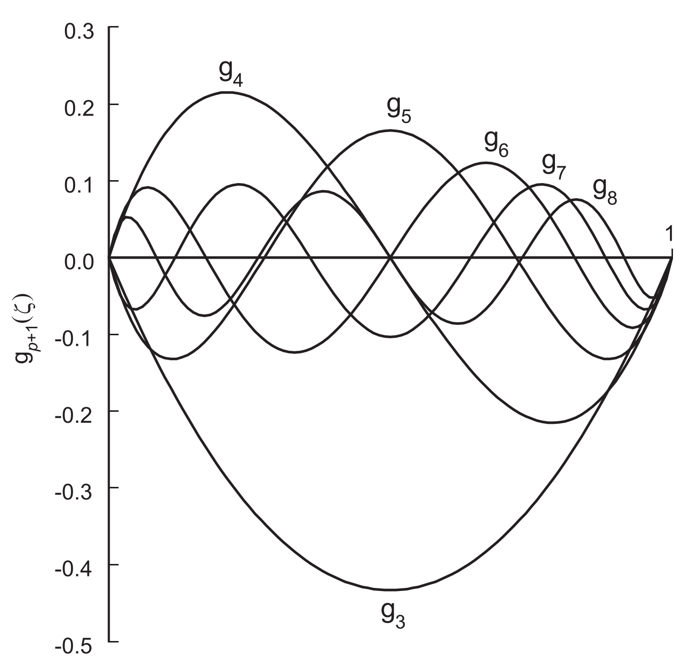

Fig. 1. The first six internal shape functions $g_{p+1}(\zeta)$ ($p = 2, 3, \ldots, 7$).