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On boundary layer in the Mindlin plate model: Levy plates

Boštjan Brank*

Faculty of Civil and Geodetic Engineering, University of Ljubljana, Jamova 2, 1000 Ljubljana, Slovenia

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Abstract

This work is related to the bending problem of thick rectangular Levy plates. Series solution for the Mindlin (thick) plate model is obtained and represented as a sum of the Kirchhoff (thin) plate model solution, the "shear terms" and the "boundary layer terms". Hard- and soft-simple supported, hard- and soft-clamped and free boundary conditions are considered. In order to detect plate regions where Kirchhoff model is good enough, and plate regions where Mindlin model should be used, a model error indicator is introduced. Several examples are presented, illustrating the difference between the Mindlin and the Kirchhoff results, the strengths of boundary layers for different boundary conditions, accuracy of several possible model error indicators and dependence of results on plate thickness.

Keywords: Plates; Mindlin plate theory; Kirchhoff plate theory; Levy plates; Boundary layer; Model error indicator

1. Introduction

The bending of plates has been modeled by different theories, which may lead to different solutions, depending on the model used. The most common plate models are the Kirchhoff model and the Mindlin model. The latter is very often referred to as the Reissner–Mindlin one, although the Reissner and the Mindlin models are somewhat different; see e.g. [1].

It is known that the Mindlin solution of the plate problem is very sensitive to the boundary conditions in the neighborhood of the boundary; the solution may vary sharply in the edge zone. This is called plate boundary layer or plate edge effect, and has been analyzed and discussed by Arnold and Falk [2,3], Häggblad and Bathe [4] and Babuška and Li [5]. The solution of the Kirchhoff model has no boundary layer. Babuška and Li [5] showed that the boundary layer is present in the solution of the three-dimensional (3d) formulation. It therefore corresponds to the physical phenomenon. Arnold and Falk [2,3] presented a theory for a rigorous analysis of the boundary layer of the Mindlin solution for plates with smooth boundary. The strengths of the boundary layers were found for different boundary conditions for rotations and stress resultants. They illustrated the theory by analyzing the exact solution of a circular and semi-infinite plate with different support conditions. Häggblad and Bathe [4] extended their work to boundary layers near a corner and made comparison of theoretical and numerical results by means of an accurate high-order plate element. Babuška and Li [5] analyzed how well the Mindlin model approximates the 3d formulation. They showed that the quality of the Mindlin solution (with respect to the 3d solution) in the neighborhood of the plate boundary strongly depends on the type of the plate boundary conditions.

The first aim of this work is to discuss the edge effects in the Mindlin solution for rectangular plates with two opposite edges hard-simply supported and the remaining two edges arbitrarily supported (e.g. hard-simple supported, soft-simple supported, hard-clamped, soft-clamped or free). Such plates are usually called Levy plates. We derive analytical (series) Mindlin solution for Levy plates, and further show that it can be represented as the sum of the corresponding Kirchhoff solution, the "shear terms" and the "boundary layer terms". So obtained Mindlin solution is then used to study and illustrate edge effects in rectangular plates for different boundary conditions.

*Tel.: +386 1 4768 594; fax: +386 1 425 0 6 93.
E-mail address: bbrank@ikpir.fgg.uni-lj.si

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We note that there are several ways to obtain closed or approximate analytical solution for rectangular plates, see e.g. [6] for a review on this topic or [7] for a series solution. Here, we exploit an approach of Lee et al. [8], Reddy and Wang [9] and Lim and Reddy [10], who derived algebraic relationships between the solutions of Mindlin and Kirchhoff plate models. In contrast with the above-mentioned works we also consider soft-simply supported and soft-clamped boundary conditions.

Mindlin-theory-based finite elements are very often used for approximate (numerical) analysis of plates. They can effectively approximate the "shear part" of the analytical solution, but they typically have problems to detect the "boundary layer part". Adaptive finite element analysis is needed to make the boundary layer effect visible, see e.g. [11–13]. In conjunction with the mesh refinement algorithm, a (mesh) discretization error indicator/estimator, which is oriented towards capturing the boundary layer effect, has to be used. We note, that the analytical Mindlin solutions, presented in this paper, can be used to estimate performance of any (mesh) discretization error indicator related to the mesh of Mindlin-theory-based finite elements; see e.g. [14,15] for examples of discretization error indicators.

The second aim of this work is related to the model error indicator, which is another source of error in the computational (numerical) plate model. It is far more difficult to estimate than the discretization error, see e.g. [16,17]. It is related to the suitability of the mathematical model chosen for the plate analysis. With the analytical solutions for Kirchhoff and Mindlin models available, a suitable model error indicator for the Kirchhoff model can be suggested. We would like to have one that is simple enough as well as sensitive enough to detect the shear layers in plate as well as the edge effects. Having this in mind, we suggest and mutually compare several model error indicators, which have a potential to detect plate regions where Kirchhoff model is fine enough and plate regions where more refined Mindlin model should be used.

The paper is organized as follows. In Section 2 we present basic equations of Kirchhoff and Mindlin plate models and algebraic relationship between those two models. We further recall basic results of theoretical edge effect analysis for Mindlin model and discuss several possible model error indicators. In Section 3 the results of Section 2 are used for the case of Levy plates. In Section 4 we present several illustrative examples. The conclusions are drawn in Section 5.

2. Plate models

In this section we present basic equations of Kirchhoff and Mindlin plate models and algebraic relationship between them. We recall basic results of edge effect analysis for Mindlin model and introduce several model error indicators.

2.1. The Mindlin and the Kirchhoff plate models and their relationship

Let us consider a plate of thickness h, which mid-plane is in the xy plane. We assume that any transverse loading on the plate can be adequately represented by $q = q(x, y)$. The three basic sets of equations of any structural model (i.e. equilibrium, kinematic and constitutive) are for the Mindlin bending plate model:

$$\begin{aligned} &Q^M_{x,x} + Q^M_{y,y} + q = 0,\\ &M^M_{xx,x} + M^M_{xy,y} - Q^M_x = 0,\\ &M^M_{yy,y} + M^M_{xy,x} - Q^M_y = 0, \end{aligned} \tag{1}$$

$$\begin{aligned} &\kappa^M_{xx} = \phi_{x,x}, \quad \kappa^M_{yy} = \phi_{y,y}, \quad 2\kappa^M_{xy} = \phi_{x,y} + \phi_{y,x},\\ &\gamma^M_x = \phi_x + w^M_{,x}, \quad \gamma^M_y = \phi_y + w^M_{,y}, \end{aligned} \tag{2}$$

$$\begin{aligned} &M^M_{xx} = D(\kappa^M_{xx} + \nu\kappa^M_{yy}), \quad M^M_{yy} = D(\nu\kappa^M_{xx} + \kappa^M_{yy}),\\ &M^M_{xy} = \left(\tfrac{1}{2}D(1-\nu)\right)2\kappa^M_{xy},\\ &Q^M_x = D^s\gamma^M_x, \quad Q^M_y = D^s\gamma^M_y. \end{aligned} \tag{3}$$

Here $M^M_{xx}, M^M_{yy}, M^M_{xy}, Q^M_x, Q^M_y$ are stress resultants, ϕ_x,ϕ_y are rotations of the fibers normal to the mid-plane (Fig. 1), w^M is deflection of the mid-plane in the z direction, $\kappa^M_{xx}, \kappa^M_{xx}, 2\kappa^M_{xy}$ are bending strains (curvatures), γ^M_x, γ^M_y are transverse shear strains, D and D^s are plate constants defined as $D = Eh^3/(12(1-\nu^2))$, $D^s = \kappa^2 Gh$, G is shear modulus $G = E/(2(1+\nu))$, E is elastic modulus, ν is Poisson's ration, κ^2 is shear correction factor usually set to 5/6 for elastic isotropic plates, and $(\circ)_a = \partial(\circ)/\partial a$. The superscript M relates a quantity with the Mindlin model. Eqs. (1)–(3) can be reorganized into three coupled differential equations in terms of w^M, ϕ_x, ϕ_y by defining the moment sum $M^M = (M^M_{xx} + M^M_{yy})/(1+\nu) = D(\phi_{x.x} + \phi_{y,y})$, see (3) and (2), and by using the constitutive and the kinematic equations in the equilibrium equations:

$$\begin{aligned} &D^s(\nabla^2 w^M + M^M D^{-1}) = -q,\\ &D^s(\phi_x + w^M_{,x}) = M^M_{,x} + \tfrac{1}{2}D(1-\nu)(\phi_{x,y} - \phi_{y,x})_{,y},\\ &D^s(\phi_y + w^M_{,y}) = M^M_{,y} - \tfrac{1}{2}D(1-\nu)(\phi_{x,y} - \phi_{y,x})_{,x}, \end{aligned} \tag{4}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

One can replace three coupled Equations (4) by a set of two uncoupled differential equations in terms of w^M and $(\phi_{x,y} - \phi_{y,x})$ as shown below. Note that expressions on the right-hand side of Eqs. $(4)_2$ and $(4)_3$ can be regarded as "equilibrium shear forces", and those on the left-hand side as "constitutive shear forces". By using both types of shear forces in the first equilibrium Equation $(1)_1$ one gets the following two equations:

$$\nabla^2 M^M = -q \Rightarrow \nabla^2(\phi_{x,x} + \phi_{y,y}) = -qD^{-1} \tag{5}$$

and

$$D^s(\nabla^2 w^M + M^M D^{-1}) = -q. \tag{6}$$