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Helmholtz evolution of a semi-infinite aquifer drained by a circular tunnel

Mohamed El Tani *

Lombardi Eng. Ltd., via Simen 19, CH-6648 Minusio-Locarno, Switzerland

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ABSTRACT

The evolution equation of a drained aquifer during the consolidation process when time is transformed into the Laplace variable is the modified Helmholtz equation. The governing equation of the steady state of a heterogeneous aquifer which hydraulic conductivity when plotted against depth in a semi-log graph has a constant slope is also the modified Helmhotlz equation. The same equation comes out when the slopes of the hydraulic conductivity plotted against depth and against the hydraulic potential in a semi-log graph are constants. The modified Helmholtz equation will be solved exactly considering a semi-infinite aquifer drained by a circular tunnel. A unique state function, which according to the case considered has different interpretations, is obtained in closed form as an infinite sum involving modified Bessel functions. The amount of water that flows into the tunnel contrarily to the state function may change from case to case and will be calculated exactly and in closed form for the different cited cases. The analytic solution has a wide range of application, is valid for different cases, and within every case needs being adapted to the particular problem to be solved. An illustrative application will show an adaptation of the solution to rock masses when the hydraulic conductivity plotted against the effective stress in a semi-log graph has a constant slope. This will allow estimating the relative precision of approximated formulae for the water inflow in fissured rock masses such as the Zhang and Franklin equation and the first order approximation.

1. Introduction

Tunneling science has grown during the computer age and has not known the same intense analytic development that well and drain theories have, as summarized by Cheng (2000). Muskat (1937) obtained the exact analytic solution of a point source and its image and deduced the water inflow equation, which was used by Goodman et al. (1965) in model experiments. Rat (1973), Schleiss (1988), and Lei (1999) used and proved the exact analytic solution of the drained steady state of a semi-infinite homogeneous aquifer considering a constant head on the tunnel edge. An exact analytic solution that considers zero pressure on the tunnel edge was presented at the world tunnel congress in Oslo (El Tani, 1999). An improved exact and compact solution that allows an arbitrary pressure or potential on the tunnel edge was then obtained using a Möbius transformation (El Tani, 2003). Complementary analytic developments, modifications for undersea tunnels and calculations of seepage forces were undertaken by Kolymbas and Wagner, 2007 and Park et al. (2008). All of these equations are solutions of the same governing equation, which is the Laplace equation. Deviating from steadiness or homogeneity makes the governing and evolution equations deviate from the Laplace equation. Finding closed analytic solutions become a harder but not impossible task. Zhang and Franklin (1993) obtained an exact analytic solution considering a point source and its image in a heterogeneous aquifer and deduced a water inflow equation for undersea tunnels. To comply with the lack of analytic solutions for transient flow, Perrochet (2005), Perrochet et al. (2007), and Hwang and Lu (2007) adapted solutions that are used in well and heat theories to construct models that predict water discharge during tunnel driving.

A unique closed analytic solution that may be applied in many different cases that involve a semi-infinite aquifer drained by a circular tunnel with an arbitrary pressure on the tunnel edge will be presented. All of these cases have in common a unique governing equation known as the modified Helmholtz equation. Four of these are:

- The transient evolution of a homogenous aquifer during the consolidation process with constant hydraulic conductivity and storage capacity.
- The steady state of a non-homogeneous aquifer where the hydraulic conductivity decreases exponentially with depth; or equivalently the slope of the hydraulic conductivity plotted against depth on a semi-log graph is constant. An example is a fractured rock mass with a decreasing number of fractures with depth.

* Corresponding author. Tel.: +41 917446030; fax: +41 0917439737.
E-mail address: mohamed.eltani@lombardi.ch

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Notations

Latin	
a	constant parameter
b	constant parameter
$\mathbf{b}$	constant vector field
C	tunnel center
C_{mn}	matrix coefficient
H	horizontal axis
h	distance of the tunnel center to H
h_{wt}	distance of the tunnel center to the water table
h_{gs}	distance of the tunnel center to ground surface or sea floor
I_n	modified Bessel of the first kind of order n
K_n	modified Bessel of the second kind of order n
S	storage capacity or the inverse of the bulk modulus
$\mathbf{K}$, K, k	hydraulic conductivity
$\mathbf{n}$	exterior normal that points upwards on H and toward C on T_x and T_y
O	origin of the coordinate axis that lies on H
$\mathbf{q}$	Darcy's velocity
Q	water flowing through closed curve
Q_0	first order approximation of the water inflow
Q_{ZF}	Zhang and Franklin water inflow
Q_{ap}	approximated water inflow
r	radius of the tunnel
r_x, r_y	radius of the circle T_x, T_y
r_x, r_{x^*}	lengths of CX, CX*
r_y, r_{y^*}	lengths of CY, CY*
S	storage capacity
S_{mn}	matrix coefficient
t	time
T_y	circle of radius $\mathrm{r_y}$ and center C
T_x	circle of radius $\mathrm{r_x}$ and center C
x_1, x_2	horizontal and vertical coordinates
V	vertical axis

Greek	
Δ	relative difference
Δ_{mn}	modified Kronecker ($\Delta_{00} = 2$, $\Delta_{\mathrm{mm}} = 1$ for $m > 0$, Δ_{mn}=0 for m different to n)
δ	dirac delta distribution
θ_x, θ_{x^*}	angles OCX, OCX*
θ_y, θ_{y^*}	angles OCY, OCY*
β	constant
θ_b	angle of $\mathbf{b}$ with the vertical axis
γ_w	water specific weight
γ_s, γ_d	saturated, dry rock specific weight
Ω	state function
φ	single layer
φ_n^c, φ_n^s	cosine and sine Fourier coefficients of φ
ψ	hydraulic head
η	hydraulic potential on T_y
ω	state function on T_y
ω_n^c, ω_n^s	cosine and sine Fourier coefficients of ω
σ^e	effective stress
σ^t	total stress
τ	Laplace variable

Operator and symbol	
$\Im$	Laplace operator
$\Im^{-1}$	inverse Laplace operator
$\bar{f}(\tau)$	Laplace transform of $f(t)$
$\mathbf{b}.\mathbf{x}$	scalar product of $\mathbf{b}$ and $\mathbf{x}$
∇	gradient operator
ϵ	belong to or is restricted to

– The steady state of a homogeneous aquifer where the hydraulic conductivity changes exponentially with the effective stress to the condition that total stress remains unchanged; or equivalently the slope of the hydraulic conductivity plotted against the hydraulic potential in a semi-log graph is constant since the change of hydraulic potential is the same as that of the effective stress. An example is a fractured rock mass with closing fissures and fractures with the increase of stress with depth.
– Zhang and Franklin (1993) found that combining the latter two cases lead also to the modified Helmholtz equation and in this case the hydraulic conductivity is the product of the corresponding hydraulic conductivities.

It should be clear that for the mentioned cases the Helmholtz governing equations appear only after some transformations. First, the transformations should be clearly stated because the Helmholtz equation will act on a unique state function that has different interpretation according to the cases considered. An integral solution to the Helmholtz modified equation considering a non-homogeneous aquifer was presented at the world tunnel congress in Oslo in 1999 and used to obtain a first order approximation. Here the complete exact closed form solution of the modified Helmholtz equation will be derived from the integral solution. This needs evaluating double integrals involving modified Bessel functions that cannot be found in the specialized referenced monographs and tables (Watson 1944; Abramowitz et al., 1972; Gradshteyn, 1996). The use of the analytic solution is straightforward only after that the necessary transformations that change the governing and evolution equations into the Helmholtz equation are clearly identified. There is an important preparation work to undertake before than the analytic solution may be used. To ease the use of the analytic solution and the flow integrals, an application concerning a drained rock mass where the hydraulic conductivity changes with depth and with the effective stress will serve as example.

In tunneling, the water that flows from the soil or the rock mass into the tunnel and the water that infiltrate from the surface through the water table have to be controlled and monitored. Many reasons may be evoked concerning tunnel construction, maintenance and exploitation, but there are two important reasons that affect the civilians. Drainage induces deformation that may damage surface constructions (Lombardi, 1992) or creates water shortage (Zhang, 1988). The amount of water flowing into the tunnel and the amount of water that is needed to recharge the aquifer are important quantities that are integrals of the gradient of the state function. These integrals, contrarily to the state function differ from case to case and will all be calculated exactly for every different case.

2. Evolution equations

A circular tunnel in an underground aquifer with a horizontal water table and an undersea tunnel with a horizontal sea floor are shown in Fig. 1. The tunnel edge T_x is a circle with center C and radius r_x. A circle T_y with center C and radius r_y greater than r_x is also shown and will be used in the computation. The center of the tunnel is on the vertical axis V and is at a distance h from the horizontal axis H. The horizontal axis merges with the water table for a tunnel in an underground aquifer and with sea floor for an undersea tunnel.