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Review

# Characterization of geodesics in the small in multidimensional psychological spaces

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## HIGHLIGHTS

- We characterized geodesics in the small in multidimensional psychological spaces.
- The distance measure in the small does not fulfill the triangle inequality.
- The geodesics in the small can be obtained from sets of tangent vectors with sum **v**.
- There exists one and only one F-face for each direction **v**.

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## ABSTRACT

Dzhafarov and Colonius (1999) proposed a theory of subjective Fechnerian distances in a continuous stimulus space of arbitrary dimensionality, where each stimulus is associated with a psychometric function that determines probabilities with which it is discriminated from its infinitesimally close neighboring stimuli. In their theory, the Finslerian metric function $F(\mathbf{x},\mathbf{v})$ plays a central role, where $\mathbf{x}$ is a point of a manifold $M$ and $\mathbf{v} \in T_{\mathbf{x}}M\setminus\{\mathbf{0}\}$ is a nonzero vector in the tangent space at $\mathbf{x}$. Dzhafarov and Colonius (2001) proved that if the Finslerian metric function $F(\mathbf{x},\mathbf{v})$ is not convex in the direction of a tangent vector $\mathbf{v}$ at $\mathbf{x}$, then there exist polygonal arcs from $\mathbf{x}$ to $\mathbf{x}+\mathbf{v}s$, with $s > 0$ sufficiently small, called Fechnerian geodesic arcs in the small for $\mathbf{v}$ at $\mathbf{x}$, whose psychometric length is strictly less than that of the straight line segment from $\mathbf{x}$ to $\mathbf{x}+\mathbf{v}s$. In their paper, the authors pointed out that: "it is important to investigate the problem of Fechnerian geodesics in the small, that is, the existence and properties of an allowable path connecting $\mathbf{x}$ to $\mathbf{y} = \mathbf{x}+\mathbf{v}s$, whose psychometric length tends to the Fechnerian distance $G(\mathbf{x},\ \mathbf{x}+\mathbf{v}s)$ as $s \to 0^+$." Consequently, the principal aim of our paper is to characterize the Fechnerian geodesic arcs in the small. We prove that the Fechnerian geodesic arcs in the small for $\mathbf{v}$ at $\mathbf{x}$ can be obtained from sets $H$ of tangent vectors at $\mathbf{x}$, provided that: (a) the sum of the vectors in $H$ is equal to $\mathbf{v}$, (b) the rays in the directions of the vectors of $H$ pass through extreme points of only one face $C_{\mathbf{x}}(\mathbf{v})$ of the convex closure of the indicatrix of $F$ at $\mathbf{x}$, and (c) the ray in the direction of $\mathbf{v}$ intersects the relative interior of $C_{\mathbf{x}}(\mathbf{v})$. Also, we prove that the Fechnerian geodesic arcs in the small for $\mathbf{v}$ at $\mathbf{x}$ determine totally their corresponding face $C_{\mathbf{x}}(\mathbf{v})$.



**Contents**



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## 1. Introduction

Fechnerian scaling is a method to compute distances between physical stimuli (such as the amplitude–frequency space of tone) from the probabilities with which each of these stimuli can be discriminated from its very close neighbors. Dzhafarov and Colonius (1999) proposed a theory of Fechnerian scaling based on a multidimensional stimulus space $M$ and on a metric function $F(\mathbf{x},\ \mathbf{u})$, named *Fechner–Finsler metric function*, where $\mathbf{x}$ is a point of $M$ and $\mathbf{u}$ is a vector in the tangent space $T_{\mathbf{x}}M \setminus \{\mathbf{0}\}$ at $\mathbf{x} \in M$. The Fechner–Finsler metric function $F(\mathbf{x},\ \mathbf{u})$ determines the *psychometric length* or *F-length* $L[C(\mathbf{a},\ \mathbf{b})]$ of any arc $C(\mathbf{a},\ \mathbf{b})$ lying in the stimulus space $M$ and connecting stimuli $\mathbf{a} = \mathbf{x}(a)$ to stimuli $\mathbf{b} = \mathbf{x}(b)$,

$$L[C(\mathbf{a},\ \mathbf{b})] := \int_a^b F(\mathbf{x}(t), \dot{\mathbf{x}}(t))dt, \tag{1}$$

where $\mathbf{x} : [a,\ b] \to M$ is a piecewise $C^1$ parametric representation of $C(\mathbf{a},\ \mathbf{b})$ with $\dot{\mathbf{x}}(t) \in T_{\mathbf{x}}M \setminus \{\mathbf{0}\}$ for all $t \in [a,\ b]$. The *Fechnerian distance* from $\mathbf{a} \in M$ to $\mathbf{b} \in M$, $G(\mathbf{a},\ \mathbf{b})$, *induced* by $F(\mathbf{x},\ \mathbf{u})$, is defined as the infimum of the psychometric lengths of all piecewise $C^1$ arcs connecting $\mathbf{a}$ to $\mathbf{b}$,

$$G(\mathbf{a}, \mathbf{b}) := \inf_{\mathbf{x} \in \Omega_{[\mathbf{a},\mathbf{b}]}} \int_a^b F(\mathbf{x}(t), \dot{\mathbf{x}}(t))dt, \tag{2}$$

where the set of all the piecewise $C^1$ arcs on $M$ is denoted by $\Omega_{[a,b]}$.

We suppose that the function $F$ satisfies the following conditions (A), (B), (C), and (D):

(A) *Finite F-compactness* of $M$: If the distances $G(\mathbf{x}_\upsilon,\ \mathbf{y})$ or $G(\mathbf{y},\ \mathbf{x}_\upsilon)$ are bounded, then the sequence $\mathbf{x}_\upsilon,\ \upsilon = 1, 2, \ldots,$ has an accumulation point.

By (A), expression (2) can be written in the following form Busemann and Mayer (1941):

$$G(\mathbf{a}, \mathbf{b}) = \min_{\mathbf{x} \in \Omega_{[\mathbf{a},\mathbf{b}]}} \int_a^b F(\mathbf{x}(t), \dot{\mathbf{x}}(t))dt, \tag{3}$$

i.e., for each $\mathbf{a},\ \mathbf{b} \in M$ the infimum in (2) is equal to the psychometric length of a certain shortest arc $C^*(\mathbf{a},\ \mathbf{b}) \subseteq M$ connecting $\mathbf{a}$ to $\mathbf{b}$, called (*Fechnerian*) *geodesic arc* from $\mathbf{a}$ to $\mathbf{b}$, i.e., $G(\mathbf{a},\ \mathbf{b}) = L[C^*(\mathbf{a},\ \mathbf{b})]$.

Dzhafarov and Colonius (1999, 2001) developed their theory of multidimensional Fechnerian scaling from three assumptions (the latest version of the generalized Fechnerian scaling can be found in Dzhafarov, 2008a,b and Dzhafarov & Colonius, 2007). We prove (Section 2) that these three assumptions imply that the function $F$ is a *metric function*, i.e., $F$ satisfies the following three conditions:

(B) $F(\mathbf{x},\ \mathbf{u})$ is continuous as function of the $2n$ variables $(\mathbf{x},\ \mathbf{u})$.

(C) $F(\mathbf{x},\ \mathbf{u})$ is positive definite, i.e., $F(\mathbf{x},\ \mathbf{u}) > 0$ for all $\mathbf{u} \neq \mathbf{0}$.

(D) $F(\mathbf{x},\ \mathbf{u})$ is positively homogeneous (of order one in $\mathbf{u}$), i.e., $F(\mathbf{x}, k\mathbf{u}) = kF(\mathbf{x},\ \mathbf{u})$ for $k > 0$.

In this paper, we consider that the stimulus space is a differentiable manifold $M$ in $R^n$, i.e., for each point $\mathbf{x} \in M$ there is some neighborhood $U$ of $\mathbf{x}$ and some integer $q$, with $0 \leq q \leq n$, such that $U$ is diffeomorphic to a subset of $R^q$. Then, the tangent space to $M$ at $\mathbf{x}$ is $R^q$, and so $R^q$ is also the tangent space to $M$ at all points $\mathbf{y} \in U$ (Spivak, 1999, p. 64). The latter allows us to define polygonal arcs from $\mathbf{x}$ to $\mathbf{x} + \mathbf{v}s$, where $\mathbf{v} \in T_{\mathbf{x}}M \setminus \{\mathbf{0}\}$, and $s > 0$ is *sufficiently small* in the sense that such arcs from $\mathbf{x}$ to $\mathbf{x} + \mathbf{v}s$ lie in $U$. These polygonal arcs, that we call *polygonal arcs* (*in the small*) *for* $\mathbf{v} \in T_{\mathbf{x}}M \setminus \{\mathbf{0}\}$ *at* $\mathbf{x} \in M$, can be constructed from a *partition-concatenation process* applied to some subset $H$ of the tangent space $T_{\mathbf{x}}M \setminus \{\mathbf{0}\}$, whenever the sum of the elements of $H$ is equal to $\mathbf{v}$. This process consists of two stages: For each tangent vector $\mathbf{u} \in H$ and for $s > 0$ sufficiently small, *partition* the vector $s\mathbf{u}$ into $n_{\mathbf{u}} \geq 1$ parts (not necessarily of the same size), and then, starting at $\mathbf{x}$, *concatenate*, in a certain order, all of these $n_{\mathbf{u}}$ parts for all $\mathbf{u} \in H$. This process will generate polygonal arcs for $\mathbf{v}$ at $\mathbf{x}$ whenever the sum of the elements belonging to $H$ is equal to $\mathbf{v}$. For the sake of simplicity, we will say "$H$ with sum $\mathbf{v}$" to mean "the sum of vectors belonging to $H$ is equal to $\mathbf{v}$".

Let $A(H)$ be the set of all the polygonal arcs in the small for $\mathbf{v}$ at $\mathbf{x}$ obtained by applying all the possible partition-concatenation processes to the set $H \subseteq T_{\mathbf{x}}M \setminus \{\mathbf{0}\}$ with sum $\mathbf{v}$. By (1) and property (D), all the arcs belonging to $A(H)$ have the same psychometric length. A set $H \subseteq T_{\mathbf{x}}M \setminus \{\mathbf{0}\}$ will be called *elemental* if it does not contain two distinct codirectional vectors ($\mathbf{w}$ and $\mathbf{u}$ are *codirectional* if $\mathbf{u} = \alpha\mathbf{w}$ for some $\alpha > 0$). Therefore, each polygonal arc in the small belongs to only one set $A(H)$, where $H$ is an elemental set. We are interested in the polygonal arcs in the small of minimal psychometric length:

**Definition 1.** A (*Fechnerian*) *geodesic arc in the small for* $\mathbf{v} \in T_{\mathbf{x}}M \setminus \{\mathbf{0}\}$ *at* $\mathbf{x} \in M$ is a polygonal arc in the small from $\mathbf{x}$ to $\mathbf{x} + \mathbf{v}s$ on $M$ with minimal psychometric length.

Equivalently, Dzhafarov and Colonius (2001) defined Fechnerian geodesic arc in the small as an arc connecting $\mathbf{x}$ to $\mathbf{x}+\mathbf{v}s$, whose psychometric length tends to the Fechnerian distance $G(\mathbf{x},\ \mathbf{x}+\mathbf{v}s)$ as $s \to 0^+$. These authors remark that it is important to investigate the existence and properties of the Fechnerian geodesic arcs in the small. In this sense, the principal aim of our work is to characterize the Fechnerian geodesic arcs in the small. As we will show, this characterization establishes useful relationships between Fechnerian geodesic arcs in the small and the shapes of the indicatrices.

We say that a set $H \subseteq T_{\mathbf{x}}M \setminus \{\mathbf{0}\}$ of tangent vectors at $\mathbf{x}$ with sum $\mathbf{v}$ is an *F-minimizing set of tangent vectors at* $\mathbf{x}$ *in the direction of* $\mathbf{v}$ if the arcs belonging to the corresponding set $A(H)$ are Fechnerian geodesic arcs in the small. Then, the characterization of Fechnerian geodesic arcs in the small is equivalent to the characterization of the $F$-minimizing set of tangent vectors.

In Section 2, we briefly present the theory of Fechnerian scaling proposed by Dzhafarov and Colonius (1999, 2001). We propose a definition of convexity of $F$ with respect to the direction of any tangent vector $\mathbf{v} \in T_{\mathbf{x}}M \setminus \{\mathbf{0}\}$ at $\mathbf{x}$. In Section 3, we prove that any $F$-minimizing set of tangent vectors at $\mathbf{x}$ with sum $\mathbf{v}$ can be expressed as a convex combination of extreme minimizing sets of tangent vectors at $\mathbf{x}$ with sum $\mathbf{v}$ (Theorem 1). In Section 4, we give some basic concepts about the convexity of sets on $R^n$, in particular, the concept of face of a convex set. In Section 5, we define the indicatrix of $F$ at $\mathbf{x}$ and give two new properties of the indicatrix: (a) a necessary and sufficient condition, in terms of the indicatrix, for the sum of the tangent vectors at $\mathbf{x}$ of a set $H \subseteq T_{\mathbf{x}}M \setminus \{\mathbf{0}\}$ to be equal to $\mathbf{v}$ (Lemma 3), and (b) for each $\mathbf{v} \in T_{\mathbf{x}}M \setminus \{\mathbf{0}\}$ there exists one and only one $F$-face of the convex closure of the indicatrix at $\mathbf{x}$ such that the relative interior of such $F$-face intersects the ray $r(\mathbf{v})$ carrying $\mathbf{v}$ (Theorem 2). This last property establishes that to