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The dynamics of decision making when probabilities are vaguely specified☆

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HIGHLIGHTS

- We consider decision making with vague probability information.
- We base our discussion on a finite version of the doubling game.
- We develop a quantum probability model for the dynamics of decision making in such a game.
- We present a way in which the model can be applied in scaled-down relevant empirical situations.
- We report empirical results, which allow a preliminary assessment of the methods and manipulations.

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ABSTRACT

Consider a multi-trial game with the goal of maximizing a quantity $Q(N)$. At each trial N, the player doubles the accumulated quantity, unless the trial number is Y, in which case all is lost and the game ends. The expected quantity for the next trial will favor continuing play, as long as the probability that the next trial is Y is less than one half. Y is vaguely specified (e.g., someone is asked to fill a sheet of paper with digits, which are then permuted to produce Y). Conditional on reaching trial N, we argue that the probability that the next trial is Y is extremely small (much less than one half), and that this holds for any N. Thus, single trial reasoning recommends one should always play, but this guarantees eventual ruin in the game. It is necessary to stop, but how can a decision to stop on N be justified, and how can N be chosen? The paradox and the conflict between what seem to be two equally plausible lines of reasoning are caused by the vagueness in the specification of the critical trial Y. Many everyday reasoning situations involve analogous situations of vagueness, in specifying probabilities, values, and/or alternatives, whether in the context of sequential decisions or single decisions. We present a computational scheme for addressing the problem of vagueness in the above game, based on quantum probability theory. The key aspect of our proposal is the idea that the range of stopping rules can be represented as a superposition state, in which the player cannot be assumed to believe in any specific stopping rule. This scheme reveals certain interesting properties, regarding the dynamics of when to stop to play.

1. Introduction

The work of William Estes has had a lasting and profound influence in the development of psychological theory. One important contribution was the instigation of a revolutionary transition from descriptive to quantitative theories, the latter presented precisely in mathematical language. A mathematical formalization of a psychological process can provide a framework within which to study what is possible and what is not, identify key theoretical issues, and provide a guide for future empirical exploration. Our aim fits such goals: we study an important theoretical problem, in the psychology of decision making, and provide a computational framework for the assumed underlying cognitive processes. We then explore the properties of the computational framework, derive an empirical prediction, and experimentally test this.

Consider a 'doubling game', in which a player starts with one unit. On each trial, the player can choose to either stop playing and take home her winnings or double her accumulated units. However, if she doubles on trial Y, she loses all and the game ends. The number of Y has been vaguely specified, e.g., a person filled a sheet with random digits, permuted them, and so produced Y (since it is impossible to write an infinite number of digits on a sheet of paper, the number Y is finite). To justify stopping on

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trial N, the player must think that Prob($N = Y$) is at least 1/2. But this is always unreasonable. The player can attempt to guess the probability distribution for Y (i.e. a Bayesian prior). However, having successfully reached trial $N - 1$ without losing, she will surely form a posterior distribution that predicts considerable probability for numbers above N. Why? Because, seeing $N - 1$ ensures that the number of digits written down were sufficiently numerous to generate $N - 1$. Knowing this makes it likely that numbers higher than $N - 1$ will be produced. Thus, the paradox in the doubling game is caused by the vague specification of Y (e.g., Bonini, Osherson, Viale, & Williamson, 1999). Note that a prior could be chosen that will delay the stopping decision to such a large trial number (e.g. 100**(100**100)) that ruin would certainly occur, but this prior is not helpful. Whichever way one attempts to specify a prior on Y, we suggest that this will not resolve the paradox.

The prescription to continue playing is paradoxical, since eventually Y will be reached. This game is a finite version of the one in St. Petersburg's paradox, in which a fair coin is tossed on every trial, tripling the accumulated payoff on obtaining e.g. heads. The problem is general, e.g., it can be recast in a single trial version and can involve payoff functions, which recommend playing, regardless of the value of Prob(next guess $= Y$) and regardless of utility functions incorporating e.g. loss aversion or diminishing returns (the latter was Bernoulli's approach in dealing with St. Petersburg's paradox).

We think that there does not now exist, and may never exist, a normative theory of rationality that could be used to provide an ideal basis for decision making. Rather, for a particular problem, people explore different perspectives of reasoning, in an attempt to identify the best course of action. When the different perspectives converge, it becomes easy to resolve the decision making problem. When they diverge, then inevitably the decision making process is more difficult. In the doubling game, a local trial perspective mandates doubling. But is this reasoning correct? A player will attempt to verify correctness by trying out other perspectives. One is a global perspective and leads the player to ask whether he will gain if he continues to play indefinitely. However, since this guarantees eventual ruin, the global perspective recommends stopping at some trial. Thus, here, the two different perspectives, local and global, are in conflict.

How does one deal with conflicting, and yet plausible, lines of reasoning (or goals, or perspectives) for a problem? Notwithstanding the normative problems, we expect that human decision makers in a doubling game will eventually stop playing. We develop a descriptive model of decision making in a doubling game, so as to provide a basis for further systematic study of this behavior. We propose to approach the problem using quantum probability (QP) theory, by which we mean the probabilistic calculus of quantum mechanics, not specifically tied to physics (e.g., Aerts & Gabora, 2005 and Atmanspacher, Romer, & Wallach, 2006). QP theory is basically a way to quantify uncertainty, which is alternative to classical, Bayesian probability (CP) theory. Predictions from QP and CP models sometimes converge, but QP theory has some properties that distinguish it from CP theory. For example, in QP theory probabilistic assessment can be strongly order and context dependence and superposition states exist, for which it is not possible to make precise statements regarding certain questions. Thus, QP models have usually been proposed to cover empirical situations for which it has been difficult to develop satisfactory CP models (e.g., Aerts & Gabora, 2005, Bruza, Kitto, Nelson, & McEvoy, 2009, Busemeyer & Bruza, 2012, Busemeyer, Pothos, Franco, & Trueblood, 2011, Khrennikov, 2010, Pothos & Busemeyer, 2009, Trueblood & Busemeyer, 2011 and Wang & Busemeyer, 2013).

One motivation for adopting QP theory here is that, traditionally, QP cognitive models have fared well in situations which appear paradoxical from a CP perspective (Busemeyer & Bruza, 2012; Pothos & Busemeyer, 2009; Wang & Busemeyer, 2013). For example, in a QP model many different lines of reasoning can co-exist in a superposition state, until a judgment is made, at which point one approach emerges to govern the decision. In a doubling game, belief regarding the next stopping rule could simultaneously incorporate biases both about large stopping rules (favoring a local perspective of the game, according to which it is advantageous to continue playing for as long as possible) and smaller stopping rules (favoring the global perspective of the game and a bias to stop). However, as we discuss, we think that many of the ideas in the present QP model can be implemented classically. Another motivation is that, because most decision making models have, until recently, focused on traditional methods (based on traditional probability measures), it is interesting to provide some groundwork for how a QP theory approach can handle decision making behavior, since, overall, both CP theory and the QP theory approaches have merits and demerits and both are worth implementing and testing in empirical research. A final motivation is that the QP model enabled the extraction of analytical solutions fairly easily.

The specification of the model allows us to consider aspects of a doubling game, which may impact on behavior. Accordingly, we next develop a simple empirical task, which tests a corresponding prediction. We note that such tasks are likely to involve considerable conceptual and methodological challenges, but their development is essential in order to appreciate how naïve observers cope with problems that are problematic from a normative perspective.

Intuition regarding behavior in the doubling game enables the following assumptions, to guide model construction. First, there is a *bias* to continue playing on any particular trial. This reflects the local perspective of the game, since the game is set up in such a way that it always makes sense to continue playing. For small trial numbers, this is because of the extremely small probability that such numbers could be Y. For larger trial numbers, as argued, if a player has already observed $N - 1$, she is unlikely to believe that Y would be N. Second, a player creates a guess for when to stop. Regardless of how such a guess is created, we assume that, when the player reaches this stopping rule, she stops playing. Finally, as the game proceeds, and obviously as long as the game has not yet ended in ruin, we assume that the player might decide to change her stopping rule. For example, if the player had decided to stop after 10 trials, and she has already won in the first nine trials, would this not make it more likely that she will continue playing? Then, a cognitive model involves a specification of how the evidence from winning successive trials impacts on confidence for the (latest) stopping rule and potentially leads to its revision.

2. Model processes and results

We first describe the model mostly conceptually and then develop the mathematical details. In the mathematical details below, a lot of the arithmetic is simple linear matrix algebra; where relevant, we make some corresponding notes, which will hopefully make the material more accessible.

We represent information about the *next* guess regarding the appropriate stopping rule, with a vector $|\psi\rangle$, called the state vector (because it describes the state), in a complex Hilbert space, such that each one of its axes is a possible stopping rule. Hilbert spaces are the vector spaces employed QP theory; they are complex vector spaces, with some convergence properties as well. Complex numbers (i.e., numbers of the form $a + i \cdot b,\ i = \sqrt{-1}$) are employed in QP theory, as a mathematical convenience. All the aspects of a system that can be observed, called, surprisingly enough, observables, are guaranteed to have real values. Also, the set of axes we employ to characterize a Hilbert space are more accurately described as basis vectors, i.e., a set of orthonormal vectors, such that any vector in the space can be expressed