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On an algebraic definition of laws

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HIGHLIGHTS

- Laws of nature can be formalized as many-sorted algebras of a special type.
- Algebraic representations of the laws form a special type of groups.
- For the case when measurement outcomes are real numbers an exhaustive classification of all possible laws can be achieved.

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ABSTRACT

An algebraic definition of laws is formulated, motivated by analyzing points in Euclidean geometry and from considerations of two physical examples, Ohm's law and Newton's second law. Simple algebraic examples constructed over a field are presented.

1. Introduction

To model a *law* with algebra we need to clarify many meanings of the word *law*. We may say that a *law* is a sort of a restriction. But obviously not any restriction is a *law*. We can also say that a law is a *stable type of relation*. But what does this mean mathematically? Is it possible to develop a rule that will indicate what type of relations can be laws and what cannot?

To begin answering these questions, Kulakov (1968, 1971) proposed a mathematical theory for the concept of a law. In subsequent years this theory was developed for the case when the relations were continuously differentiable functions on smooth manifolds (Mikhailichenko, 1972). Here, these ideas are developed using an algebraic approach.

Geometry

To introduce the problem, let us consider some examples from geometry. Consider the finite set $\mathfrak{M} = \{i_1, i_2, \ldots, i_n\}$, consisting of n arbitrarily located points on a Euclidean plane. Can we say that with the arbitrary location of points there exists a particular law that relates all points of the set $\mathfrak{M}$? We have to look at all possible pairs of points of $\mathfrak{M}$ to answer this question. The number of unordered pairs is $\frac{1}{2}n(n-1)$. For each pair we use the numerical distance between them measured with a ruler to characterize their relative positions. It is assumed that measurement of the distances is exact.

Assigning the distance ℓ_{ik} to each pair of points (ik), we have a set of data obtained from the experiment which fully describes the properties of the set $\mathfrak{M}$. We can present this data set as a symmetric matrix:

$$\begin{array}{c|ccccc} & i_1 & i_2 & i_3 & \ldots & i_n \\ \hline i_1 & 0 & \ell_{12} & \ell_{13} & \ldots & \ell_{1n} \\ i_2 & \ell_{12} & 0 & \ell_{23} & \ldots & \ell_{2n} \\ i_3 & \ell_{13} & \ell_{23} & 0 & \ldots & \ell_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ i_n & \ell_{1n} & \ell_{2n} & \ell_{3n} & \ldots & 0 \end{array}$$

It is clear that the distances ℓ_{ik}, ℓ_{im}, ℓ_{km} between any *three* points $i, k, m \in \mathfrak{M}$ cannot satisfy any functional dependence, because if the distances ℓ_{ik} and ℓ_{im} are fixed, the third distance ℓ_{km} can take

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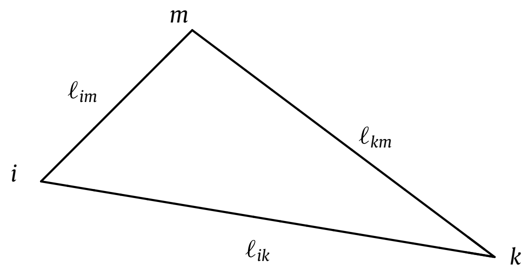
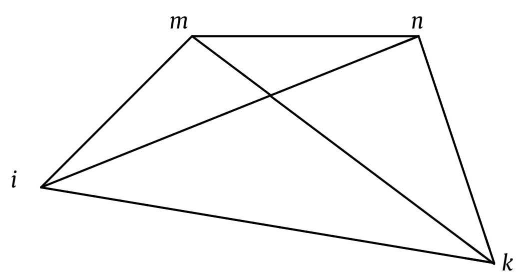
the values from $|\ell_{ik} - \ell_{im}|$ to $\ell_{ik} + \ell_{im}$.



But if we take any *four* points $i, k, m, n \in \mathfrak{M}$, then one of the *six* relative distances $\ell_{ik}, \ell_{im}, \ell_{in}, \ell_{km}, \ell_{kn}, \ell_{mn}$ is a two-valued function of the other five.



So, for every four points of the Euclidean plane there exists a functional dependence between their relative distances, which does not depend on the choice of points:

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & \ell_{ik}^2 & \ell_{im}^2 & \ell_{in}^2 \\ 1 & \ell_{ik}^2 & 0 & \ell_{km}^2 & \ell_{kn}^2 \\ 1 & \ell_{im}^2 & \ell_{km}^2 & 0 & \ell_{mn}^2 \\ 1 & \ell_{in}^2 & \ell_{kn}^2 & \ell_{mn}^2 & 0 \end{vmatrix} = 0.$$

If the four points were allowed to lie in the three-dimensional space, this determinant would be proportional to the volume of the simplex they would form. If we have zero three-dimensional volume, than all four points lie on the same plane.

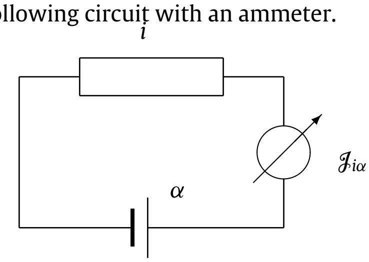
Generalizing the previous example, we can take two sets of points $i, k, m, n \in \mathfrak{M}$ and $\alpha, \beta, \gamma, \delta \in \mathfrak{M}$ of the Euclidean plane $\mathfrak{M}$ and consider the relative distances between the sets of points with Greek and Latin indexes. For any sets of points there exists a functional dependence between their relative distances, which is expressed by the Cayley–Menger determinant being zero (Kulakov, 1995).

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & \ell_{i\alpha}^2 & \ell_{i\beta}^2 & \ell_{i\gamma}^2 & \ell_{i\delta}^2 \\ 1 & \ell_{k\alpha}^2 & \ell_{k\beta}^2 & \ell_{k\gamma}^2 & \ell_{k\delta}^2 \\ 1 & \ell_{m\alpha}^2 & \ell_{m\beta}^2 & \ell_{m\gamma}^2 & \ell_{m\delta}^2 \\ 1 & \ell_{n\alpha}^2 & \ell_{n\beta}^2 & \ell_{n\gamma}^2 & \ell_{n\delta}^2 \end{vmatrix} = 0.$$

Ohm's law

In the geometry example just described, all points belong to the single set $\mathfrak{M}$. Ohm's law provides a different example where points from two different sets are matched by the result of a measurement procedure, an analog to the *distance*. (The measured values do not satisfy the triangle inequality. It is just an analogy.)

Consider the set of resistors $\mathfrak{M}$ and the set of voltage sources $\mathfrak{N}$. For any $i \in \mathfrak{M}$ and $\alpha \in \mathfrak{N}$ let us measure the electrical current in the following circuit with an ammeter.



In this case the ammeter indication $\mathcal{J}_{i\alpha}$ is an analog of the *distance* between the resistor i and the voltage source α. Consider three independent resistors $i, k, m \in \mathfrak{M}$ and two optional voltage sources $\alpha, \beta \in \mathfrak{N}$. Let us measure the *six* ammeter outputs $\mathcal{J}_{i\alpha}, \mathcal{J}_{i\beta}, \mathcal{J}_{k\alpha}, \mathcal{J}_{k\beta}, \mathcal{J}_{m\alpha}, \mathcal{J}_{m\beta}$. Assuming exact measurements, we have (Kulakov, 1968):

$$\begin{vmatrix} 1 & \mathcal{J}_{i\alpha}^{-1} & \mathcal{J}_{i\beta}^{-1} \\ 1 & \mathcal{J}_{k\alpha}^{-1} & \mathcal{J}_{k\beta}^{-1} \\ 1 & \mathcal{J}_{m\alpha}^{-1} & \mathcal{J}_{m\beta}^{-1} \end{vmatrix} = 0. \tag{1}$$

Using the reference points $k, m \in \mathfrak{M}$, $\beta \in \mathfrak{N}$, we can obtain the well-known Ohm's law for the whole circuit (Kulakov, 1968)

$$\mathcal{J}_{i\alpha} = \frac{\mathcal{E}_\alpha}{R_i + r_\alpha},$$

where $\mathcal{E}_\alpha$ is an electromotive force, r_α is the inner resistance of the voltage source α and R_i is the resistance of the resistor i.

Newton's second law

Consider Newton's second law $f = ma$, where m is the mass of the body, a is the body's acceleration and f is the driving force applied to the body. Difficulties arise when we try to understand and define the concepts of mass and force which this law contains. Mass is a measure of inertia, but this definition is implicit in the law itself. What is a force? Force – according to Lagrange – is a reason for the body's movement or a reason which intends to move. Consider the traditional statement of Newton's second law: "The driving force on a particle is equal in value and direction to the product of the material point acceleration and its mass in an inertial reference frame". Here the non-trivial concept of an *inertial reference frame* is introduced and three physical values are linked, two of which have not been defined. Is it possible to formulate Newton's second law in such a way, that does not require a definition for mass and force?

Consider two sets: the set of bodies $\mathfrak{M}$ and the set of force sources (or accelerators) $\mathfrak{N}$. One body and one force source can be paired to change the speed. We can measure such a change as acceleration $a_{i\alpha}$ of a body $i \in \mathfrak{M}$ under the applied force $\alpha \in \mathfrak{M}$.

In this case, acceleration $a_{i\alpha}$ is an analog of the *distance* between the body i and the force source α. Consider any two bodies $i, j \in \mathfrak{M}$ and any two force sources $\alpha, \beta \in \mathfrak{N}$ and measure *four* accelerations $a_{i\alpha}, a_{i\beta}, a_{j\alpha}, a_{j\beta}$. Assuming exact measurements, we have:

$$\begin{vmatrix} a_{i\alpha} & a_{i\beta} \\ a_{j\alpha} & a_{j\beta} \end{vmatrix} = 0, \tag{2}$$

by which, using the gauge points $j \in \mathfrak{M}$, $\beta \in \mathfrak{N}$, we have Newton's second law (Kulakov, 1968):

$$F_\alpha = m_i a_{i\alpha}.$$

2. Formalization

We set up the following definitions. *An algebraic system or algebra* $\langle G; \sigma \rangle$ is a set G (basic set) with the operations set σ (signature),