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Geometric–optical illusions and Riemannian geometry

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HIGHLIGHTS

- Geometric–optical illusions arising from target–context stimulus interactions are studied.
- The case of circle targets is treated in a general framework of Riemann geometry.
- Perceptual distortions are modeled as context-induced perturbations of the base geometry.
- Numerical methods to compute the shape of the distorted percept are provided.
- Magnitudes of the distortions are measured in a psychophysical experiment.

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ABSTRACT

Geometric–optical illusions (GOI) are a subclass of a vast variety of visual illusions. A special class of GOIs originates from the superposition of a simple geometric figure ("target") with an array of non-intersecting curvilinear elements ("context") that elicits a perceptual distortion of the target element. Here we specifically deal with the case of circular targets. Starting from the fact that (half)circles are geodesics in a model of hyperbolic geometry, we conceive of the deformations of the target as resulting from a context-induced perturbation of that "base" geometry. We present computational methods for predicting distorted shapes of the target in different contexts, and we report the results of a psychophysical pilot experiment with eight subjects and four contexts to test the predictions. Finally, we propose a common scheme for modeling GOIs associated with more general types of target curves, subsuming those studied previously.

1. Introduction

Visual perception informs us about the outward reality in the surrounding space. Under certain circumstances, the result of a perceptual process ("percept", for short) may remarkably differ from our knowledge of the objective reality as it is evidenced, e.g., by measurement results, cognitive inferences, or other percepts. Those situations are commonly known as "visual illusions".[1] Visual illusions are not deliberate deceptions or random errors of the visual system; they are systematically occurring, experimentally reproducible and measurable phenomena, presumably revealing essential properties of the visual system, and as such they are a proper subject of scientific study (Coren & Girgus, 1978; Eagleman, 2001; Metzger, 1975; Robinson, 1998). Following Gregory's (1997b) proposal, visual illusions can be roughly subdivided into four types: fictions, paradoxes, ambiguities, and distortions.

Geometric–optical illusions (GOI) are an interesting subclass of visual *distortions*. In GOIs, geometric properties of a stimulus – e.g., lengths, angles, areas, or forms – are affected and systematically altered by the presence of other elements in the visual field. For example, a straight line appears slightly curved when superposed

Abbreviations: GOI, geometric–optical illusion; LI, local interactions; PDF, Portable Document Format; PM, Poincaré model; RO, regression to orthogonality; VHI, vertical–horizontal illusion.

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[1] Of course, perceptual illusions are known also in other sensory modalities, or occurring as inter-modal interactions. However, these types of illusions are not in our focus, nor are visual illusions affecting optical qualities such as brightness or color, and phenomena involving illusory motion. An exhaustive overview of the field is beyond the scope of the present paper.

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with an array of straight or curved lines (Hering, 1861); a length marked by two distinct elements appears larger if the space between them is subdivided by additional elements (Kundt, 1863; Oppel, 1861); two equally long line segments appear different when marked by arrows of opposite orientation (Müller-Lyer, 1889, probably the most popular GOI); etc. These phenomena were discovered and named about one and a half century ago (Oppel, 1855), and since then their number significantly expanded. Despite numerous classificatory (Coren, Girgus, Erlichman, & Hakstian, 1976; Gregory, 1997b) and explanatory attempts (Changizi, Hsieh, Nijhawan, Kanai, & Shimojo, 2008; Coren & Girgus, 1978) based on optical, retinal, cortical or cognitive mechanisms, there is by now no unitary theory of GOIs, let alone of visual illusions on the whole. There is not even consensus about general principles upon which such a theory could or should be based (Wackermann, 2010; Zavagno, Daneyko, & Actis-Grosso, 2015). Nonetheless, the GOIs deserve special attention: not only because, historically, they "form the core of the subject [of visual illusions]" (Robinson, 1998, p. 11), but also for their link to geometry-based theories of visual perception.

In the present paper we aim at a *phenomenological* theory of GOIs ; that is, we search for a mathematical representation of the phenomena under study, not for an explanation via physiological or psychological mechanisms. Our focus is on a special class of GOIs based on interactions between a *target* element and *context* elements in the visual field. In our previous work (Ehm & Wackermann, 2012) we were studying Hering type illusions, where the target was a segment of a straight line. Here we consider the case where the target is a *circle*, such as in Fig. 1(a), (b). These and similar illusions were described, independently and in different conceptual frameworks, by Ehrenstein (1925) and Orbison (1939). Illusions of this type and of Hering type have in common the *angular expansion* effect, also known as "regression to right angles" (Hotopf & Ollerearnshaw, 1972; Hotopf & Robertson, 1975): the illusory distortion of the target acts so as to enlarge the acute angles at the intersection points. It is thus plausible to assume that a common approach may account for both groups of phenomena.

In our earlier paper (Ehm & Wackermann, 2012) we modeled the distorted percepts of a straight line segment (target) by the solutions of a variational problem, namely as the shortest path connecting the endpoints of the target when length is measured in terms of a context-induced perturbation of the Euclidean metric. In regard to the circular targets in Ehrenstein–Orbison type illusions the questions arise: can this principle be generalized? And, if so, how to overcome the restriction to straight-line targets inherent in the original approach? The basic idea permitting an extension to curved targets consists in *equipping the target itself with a geometry* (Ehm & Wackermann, 2013). In this view both the target and the distorted percepts figure as paths of shortest length (geodesics) in an appropriate "base" geometry and a context-induced perturbation of that geometry, respectively. In the present paper we elaborate on this approach, using the fact that half circles represent geodesics in a suitable model of hyperbolic geometry.

The elements of this framework are presented in Section 2. We introduce the base and the perturbed hyperbolic geometries and describe the single steps leading to our final prediction of the distorted percept. The details are deferred to the mathematical Appendices. An experiment intended to verify the predictions and to measure the magnitude of the illusory distortion is reported in Section 3. The discussion in the final Section 4 addresses phenomenological as well as modeling aspects. It concludes with the above indicated proposal for a general approach covering GOIs of the Hering and the Ehrenstein–Orbison type as special cases.

2. Mathematical model

2.1. Preliminary remarks

Our mathematical description of the visual distortion of the target figure draws on minimum principles related to geometrical conceptions. We think of the context figure as distorting the spatial relationships between the points of the drawing plane, similar to when an elastic substance is kneaded: some portions are expanded while others are condensed. Riemannian geometry (Laugwitz, 1965) makes it possible to describe such situations mathematically by means of a locally varying metric that attaches a well-defined length to every path through the respective range. The connection to minimum principles comes via the concept of a *geodesic*, which is a path of minimal length, measured in the respective metric, that connects two given points. In fact, geodesics also represent paths requiring minimum energy or effort, rendering them relevant to fundamental conceptions about human perception; cf. Section 4.3.

According to our basic surmise the percept of the circular target can be modeled as a geodesic in a suitable geometry that is perturbed by the context if such is present. It is known that a (2D) Riemannian geometry admitting only circles and straight lines as geodesics must have constant curvature (Khovanskii, 1980). The possible candidates for a "suitable" or *base* geometry thus must be one of the classical elliptic, Euclidean, and hyperbolic geometries. Among these only Poincaré's half plane model of hyperbolic geometry [PM] fulfills the following three conditions essential to our approach.[2]

(i) The base geometry has to have circle segments as geodesics—so does the PM.
(ii) The perceptual distortions are hypothesized to obey the local interactions principle, particularly, to depend on the intersection angles between the context and the target lines. Therefore, the intersection angles should be represented faithfully—the PM is conformal.
(iii) The relevance of geometries derived from immersions of a surface into $\mathbb{R}^3$ appears doubtful considering that our stimulus figures are presented, and seen, on a flat screen—the PM implements the distortions of the Euclidean metric entirely within (a subset of) $\mathbb{R}^2$.

To summarize, adopting the PM for the base geometry is virtually cogent for our modeling approach. The PM allows us to deal with circular targets essentially along the same lines as with the straight line targets considered in the earlier work (2012), although the details are rather more involved. In this section we only present the main lines of our geometrical approach; the mathematical elaboration is postponed to the Appendices.

2.2. Poincaré model of the hyperbolic plane

The Poincaré model equips the upper half plane $\mathscr{H} = \{\xi = (\xi_1, \xi_2) \in \mathbb{R}^2,\ \xi_2 > 0\}$ with the line element $ds^2 = (d\xi_1^2 + d\xi_2^2)/\xi_2^2$. (For comparison, the Euclidean line element is $d\xi_1^2 + d\xi_2^2$.) This means that the length of a smooth curve $t \mapsto x(t) = (x_1(t), x_2(t))$, $t_0 \le t \le t_1$ in $\mathscr{H}$ is, invariantly under reparameterization, defined as

$$L(x) = \int_{t_0}^{t_1} \frac{|\dot{x}(t)|}{x_2(t)}\, dt = \int_{t_0}^{t_1} \sqrt{\langle \dot{x}(t), H(x(t))\, \dot{x}(t)\rangle}\, dt. \qquad (1)$$

[2] Strictly speaking, Poincaré's disc model (Cannon, Floyd, Kenyon, & Parry, 1997) satisfies (i) to (iii), too. However, it serves our purposes less well as the geodesics span less than a half circle, and the contexts have to be placed in the disc; both circumstances imply substantial restrictions.