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Height reversal generated by rotation around a vertical axis

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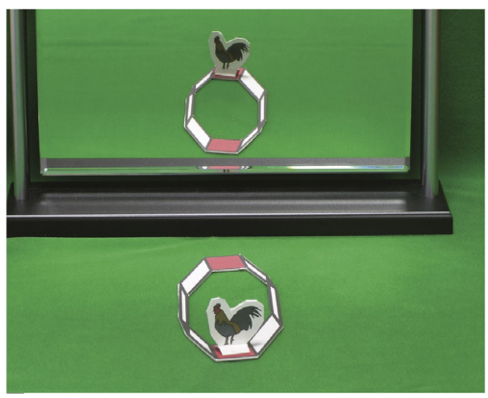
HIGHLIGHTS

- Height reversal property common to pictures drawn on a horizontal plane.
- Height reversal generated by 3D rotation instead of 2D rotation.
- Anomalous objects that change the appearances in a mirror.

GRAPHICAL ABSTRACT



The 3D height of a ring is reversed if it is reflected in a mirror (i.e., if it is rotated by 180 degrees around a vertical axis). As a result, a rooster standing at the bottom jumps up on top of the ring.

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ABSTRACT

This paper presents a mathematical framework for explaining "height reversal", a class of depth-reversal phenomena that occur when interpreting images. In particular, it is proved that, if a picture on a horizontal plane evokes an impression of depth for a viewer who sees it from an oblique direction, then when that same picture is rotated by 180° around a vertical axis, it evokes an impression of reversed height. Visual effects caused by this 3D rotation are different from the 2D rotation of turning a picture upside down, because additional objects outside the horizontal plane are also rotated. Examples of height-reversing scenes are constructed, and their relations with known depth-reversal phenomena are discussed.

1. Introduction

An optical illusion is a phenomenon in which what we perceive is different from physical reality. Such a phenomenon is not only interesting in itself, but also important in vision science, because it can be used to obtain experimental data about extreme behaviors of human vision systems (Kitaoka, 2010; Robinson, 1998).

One typical class of optical illusion is "depth reversal", in which the depth we perceive for an object is different from physical reality, or in which two depths are perceived alternately. Depth-reversal illusions are subdivided into several types.

The first type is evoked by ambiguous pictures, such as a Necker cube and a Mach book, in which observers alternately perceive two opposite interpretations of depth, even though they are viewing the same still picture without changing viewpoint (Gregory, 1970; Robinson, 1998; Shepard, 1990; Unruh, 2001).

The second type is the crater illusion (Adams, 2008; Ramachandran, 1988; Schofield, Rock, & Georgeson, 2011), in which the perceived depth is reversed when the picture is turned upside down. This illusion can be explained by the tendency of the human brain to interpret shade information by assuming that the scene is illuminated from above or left above (Adams, 2008; Mamassian & Landy,

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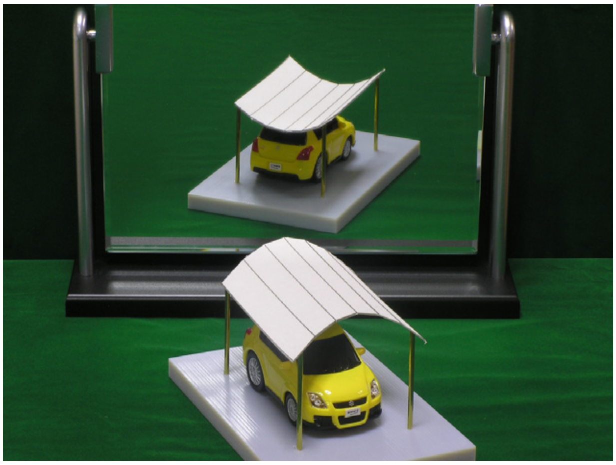

Fig. 1. Height reversal of a roof.

2001; Ramachandran, 1988; Schofield et al., 2011). The same type of illusion can be created by illuminating rough walls from below.

The third type is the hollow mask illusion (Gregory, 1970). In this illusion, a mask of a human face is perceived to be convex even if the back side of the mask is shown. This illusion is explained by the fact that we have a prior knowledge that faces are convex.

In the fourth type of depth-reversal illusion, a line drawing (i.e., a picture without shading information) generates a reversal of depth when it is turned upside down (Mamassian & Landy, 1998).

There is, however, still another type of depth-reversal phenomenon. Suppose that a picture is drawn on a horizontal plane and that it is viewed from above in a skew direction. If this picture evokes an impression of height, the reverse height will be evoked when it is rotated by 180° around a vertical axis. We call this phenomenon "height reversal".

An example is shown in Fig. 1. In this photograph, there is a vertical mirror behind a garage. Note that the roof in the mirror appears to be a reversal of the direct view of the roof in the vertical direction. We will show that this phenomenon can be explained mathematically in the sense that the rotation around a vertical axis by 180° generates a retinal image corresponding to the reversed height. On the basis of this mathematical property, we also present a method for constructing pictures that generate a height-reversal illusion.

2. Theory

As shown in Fig. 2, let us fix an *xyz* coordinate system in such a way that the *xy* plane is horizontal and consequently the *z* axis is vertical. Suppose that a picture is drawn on the *xy* plane and that we see it from two locations on the *xz* plane, both at an angle of θ above the horizontal. In other words, we see the picture from the two viewpoints defined in the following way.

We consider two viewpoints E_1 and E_2, both included in the *xz* plane and both above the *xy* plane. More specifically, let E_1 be the point at infinity in the direction forming angle θ with respect to the positive *x* direction, and let E_2 be the point at infinity in the direction forming angle θ with respect to the negative *x* direction. We call θ the "pitch angle".

As shown in Fig. 3, suppose that we see a three-dimensional object from viewpoint E_1 and project the object onto the *xy* plane, obtaining a picture of the object. Let $P = (x, y, z)$ be an arbitrary point on the object, and let Q be the corresponding point on the *xy* plane. Let $P' = (x, y, -z)$ be the point that is mirror symmetric to P with respect to a mirror placed on the *xy* plane. We can prove that P' is on the line connecting E_2 and Q. In other words, if we see the object and its projection from E_2, Q and P' coincide in the retinal image. Thus, we get the next proposition.

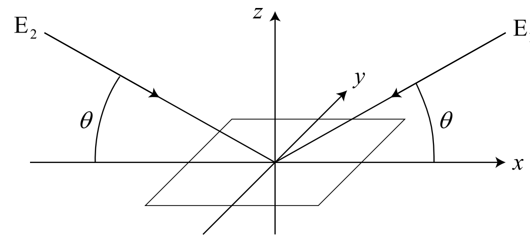

Fig. 2. Picture drawn on the horizontal plane and viewed from two vertically symmetric directions.

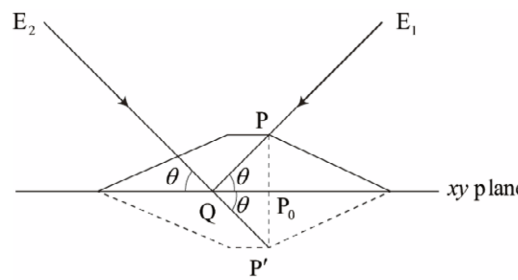

Fig. 3. Height reversal property.

Proposition 1. *Let* $P = (x, y, z)$ *be any point in three-dimensional space, and let* $P' = (x, y, -z)$ *be its mirror image with respect to the xy plane as the mirror. Then, line* E_1P *and line* E_2P' *intersect at a point on the xy plane.*

Proof. Because E_1 and E_2 are points at infinity, the lines E_1P and E_2P' are both parallel to the *xz* plane. Because P and P' have the same *y* coordinate, line PP' is also parallel to the *xz* plane. Consequently, the three lines E_1P, E_2P', and PP' are all on the same plane parallel to the *xz* plane. Hence, the lines E_1P and E_2P' intersect. Let us denote this point of intersection by Q. Let P_0 be the midpoint of P and P'. Because P and P' are symmetric with respect to the *xy* plane, P_0 is on the *xy* plane. Because both the angle PQP_0 and the angle $P'QP_0$ are equal to the pitch angle θ, the triangle PQP_0 and the triangle $P'QP_0$ are congruent, and consequently, line P_0Q is perpendicular to line PP', which implies that Q is on the *xy* plane. □

This proposition implies the following.

Proposition 2 (*Height Reversal Theorem*)**.** *Suppose that S is a surface made of opaque material and that the whole part of S is visible from* E_1*. Let S′ be the mirror image of S with respect to the xy plane as the mirror. Then the projection of S onto the xy plane with respect to the direction* E_1 *of the projection coincides with the projection of S′ onto the xy plane with respect to the direction* E_2 *of the projection. In particular the whole surface S′ is visible from* E_2*.*

Proof. The main part has already been proved by Proposition 1. The visibility of S' from E_2 is guaranteed because the whole surface S is visible from E_1, hence the picture of S on the horizontal plane represents the whole part of S, and this picture is visible from E_2. □

Therefore, if we draw a picture on the *xy* plane appropriately and view it from both E_1 and E_2, we can expect that we will perceive two surfaces that have opposite heights (vertical distances from the *xy* plane). In other words, if we fix the viewpoint at E_1 and rotate the picture around the *z* axis by 180°, then we can expect that the perceived height will be reversed. Thus, Proposition 2 gives us a basic principle for designing a height-reversal illusion.