Journal of Mathematical Psychology 68–69 (2015) 37–48

Contents lists available at ScienceDirect

Journal of Mathematical Psychology

journal homepage: www.elsevier.com/locate/jmp

On mimicry among sequential sampling models

Arash Khodadadi *, James T. Townsend

Indiana University, Department of Psychological and Brain Sciences, Bloomington, IN, United States

HIGHLIGHTS

- We prove the mimicry of a Wiener process by an independent race model.
- We examine the numerical computation of the mimicking boundaries.
- We show that the mimicking boundaries are time-varying and asymmetric.
- We propose an equivalent symmetric race model.
- We examine the mimicry of full diffusion model.

ARTICLE INFO

Article history:
Received 16 March 2015
Received in revised form
5 August 2015
Available online 19 November 2015

Keywords:
Sequential sampling models
Wiener process
Independent race model
Inverse first passage time problem
Model mimicry

ABSTRACT

Sequential sampling models are widely used in modeling the empirical data obtained from different decision making experiments. Since 1960s, several instantiations of these models have been proposed. A common assumption among these models is that the subject accumulates noisy information during the time course of a decision. The decision is made when the accumulated information favoring one of the responses reaches a decision boundary. Different models, however, make different assumptions about the information accumulation process and the implementation of the decision boundaries. Comparison among these models has proven to be challenging. In this paper we investigate the relationship between several of these models using a theoretical framework called the inverse first passage time problem. This framework has been used in the literature of applied probability theory in investigating the range of the first passage time distributions that can be produced by a stochastic process. In this paper, we use this framework to prove that any Wiener process model with two time-constant boundaries can be mimicked by an independent race model with time-varying boundaries. We also examine the numerical computation of the mimicking boundaries. We show that the mimicking boundaries of the race model are not symmetric. We then propose an equivalent race model in which the boundaries are symmetric and time-constant but the drift coefficients are time-varying.

1. Introduction

In the last few decades, a large amount of research has investigated the mechanisms underlying simple perceptual decision making. The basic idea is to examine how the subjects' reaction time and accuracy change as a function of the properties of noisy stimuli. Describing the pattern of this empirical data computationally has proven to be a challenging task. A "good" computational model should be able to describe the relation between the physical properties of the stimulus (e.g., the salience and discriminability) and the shape of the reaction time distributions, the accuracy, the relative speed of the correct and incorrect responses and the effect of emphasizing speed or accuracy in the instructions. Neurophysiological data obtained from the activity of populations of neurons during perceptual decision making experiments impose more restrictions on the computational models. One class of models which has been successful in accounting for these patterns of data is *sequential sampling models*. In this modeling framework, it is assumed that after the presentation of the stimulus, the subject starts accumulating noisy information favoring each alternative response in the task. The subject responds in a trial when the accumulated information favoring one of the alternatives reaches a specific amount called the *decision threshold*.

Several instantiations of this framework have been proposed by researchers including the full diffusion model (Ratcliff, 1978), Ornstein–Uhlenbeck (OU) model (Busemeyer & Townsend, 1993), leaky competing accumulator (LCA) model (Usher & McClelland,

* Correspondence to: Indiana University, Department of Psychological and Brain Sciences, 1101 E. 10th street, 47405-7007, Bloomington, IN, United States.
E-mail address: arakhoda@indiana.edu (A. Khodadadi).

http://dx.doi.org/10.1016/j.jmp.2015.08.007
0022-2496/

2001), linear ballistic accumulator (LBA) model (Brown & Heathcote, 2008), race models (Eidels, Houpt, Altieri, Pei, & Townsend, 2011; Smith & Vickers, 1988; Townsend & Ashby, 1983) and accrual halting models (Townsend, Houpt, & Silbert, 2012). These models differ in their assumptions about the information accumulation process and the way that the decision is made based on this information.

Comparison among these models poses another challenge to the computational modeling of perceptual decision making. Model comparison is particularly challenging because in many situations these models make similar predictions. Two general approaches have been employed by researchers to compare these models. In the first approach, the models are fitted to the empirical data and are compared based on some statistical measures of *goodness of fit*, for example, chi-square, sum of squared errors, BIC and AIC (Ratcliff & Smith, 2004; Ratcliff & Tuerlinckx, 2002; Ratcliff, Van Zandt, & McKoon, 1999; Tsetsos, Gao, McClelland, & Usher, 2012; Van Zandt, Colonius, & Proctor, 2000). Besides quantitative fit, the qualitative predictions of each model are compared to the patterns in the data. For example, a common finding in reaction time experiments is that the mean reaction times for the correct and incorrect responses are not the same. Any model that cannot predict this pattern is not likely to be a good model of these experimental data.

The second approach is to examine the theoretical relationship between these models without considering the data (Dzhafarov, 1993; Jones & Dzhafarov, 2014; Pike, 1968; Smith, 2010; Townsend, 1976; Townsend & Ashby, 1983; Zhang, Lee, Vandekerckhove, Maris, & Wagenmakers, 2014). Of particular interest in this vein of research is the problem of *model mimicry*. Two models of the reaction time mimic each other if they produce the exact same distributions of reaction time. The research in this area is less prevalent. One main reason is that, besides a few exceptions, the analytic form of the distributions of reaction time predicted by these models is not known. Therefore, it is hard to determine the range of patterns that can be produced by each model. This is especially the case when the accumulation process is modeled as a stochastic process. Recently, Jones and Dzhafarov (2014) have theoretically investigated the range of reaction time distributions that can be produced by several classes of models. In the models considered in their paper, neither the accumulation process nor the decision thresholds are stochastic processes. Instead, the models consist of random variables and deterministic time-varying functions.[1] More recently, Zhang et al. (2014) proposed a new method for investigating the mimicry between sequential sampling models in which the accumulation process is a stochastic process and the decision thresholds are time-varying functions. In their method, the problem of mimicking a model by another model is translated into another problem called *the inverse first passage time problem*. A model can mimic another one if the corresponding inverse first passage time problem is solvable. The authors considered the mimicry between a diffusion model and an accumulator model. They showed how one can numerically compute two time-varying boundaries for a diffusion model such that it mimics an accumulator model with symmetric boundaries. Although their simulation results suggest that an accumulator model can always be mimicked by a diffusion model, no theoretical analysis is provided in the paper.

In this paper, we take the same approach for investigating mimicry among sequential sampling models. Specifically, we consider the following question: can a Wiener process with constant boundaries be mimicked by an independent race model? The main goal of this paper is to investigate this question theoretically using the existing theorems in the stochastic processes literature, particularly the inverse first passage time problems. To this end, in the following two sections we introduce the stochastic processes considered in this paper and give a formal definition of the inverse first passage time problem. Then, in Section 4 we review some of the existing theorems regarding the inverse first passage time problem that we will use in deriving our results. In Section 5, we present our theoretical results on mimicry between the Wiener process model and the independent race model and then in Section 6 the numerical results are reported. In Section 7, we compare our results with some of the theoretical results in Jones and Dzhafarov (2014). Finally, Section 9 is devoted to the problem of mimicry of a Wiener process model by an OU process model.

2. Wiener process and independent race models of decision making

As explained in the Introduction, in a sequential sampling model it is assumed that the information favoring each alternative is accumulated and the subject responds in a trial whenever the accumulated information reaches a decision boundary. In a Wiener process model (also known as the Wiener diffusion model) of a task with two alternatives, the accumulated information is modeled as a stochastic process called the Wiener process. Formally, a Wiener process $X(t)$ is characterized by the following stochastic differential equation (SDE):

$$dX(t) = \mu \cdot dt + \sigma \cdot dB(t). \tag{1}$$

In this equation, the parameters μ and σ are called the drift and the diffusion coefficients, respectively. It can be shown that $\mathrm{E}[\mathrm{X(t)}] = \mu \cdot t$ and $\mathrm{Var}[\mathrm{X(t)}] = \sigma^2 \cdot t$ and so these parameters determine the mean and the variance of the process at each time (see for example Smith, 2000). The process dB specifies the increments of a zero-mean Gaussian process. In this paper, we always assume that the initial value of the process is zero ($X(0) = 0$).

In this model, it is assumed that the response 1 (response 2) is chosen in a trial if the process exceeds the decision boundary b_1 (b_2) before it hits the other decision boundary b_2 (b_1). In the literature of the stochastic processes, the first time that the process hits a decision boundary is called the first passage time (FPT). In the sequential sampling models, the FPT of the process is considered as the subject's decision time. Formally, the FPTs for the decision boundaries b_1 and b_2 are defined as follows:

$$\begin{aligned} T_1 &= \inf\{t > 0 | X(t) \geq b_1 \text{ AND } X(\tau) > b_2, \text{ for all } \tau < t\} \\ T_2 &= \inf\{t > 0 | X(t) \leq b_2 \text{ AND } X(\tau) < b_1, \text{ for all } \tau < t\}. \end{aligned} \tag{2}$$

Because of the noise term dB in Eq. (1), the FPTs T_i are random variables. The subject's reaction time in each trial is a realization of either of these two random variables. We assume that when the accumulated information of one accumulator reaches its threshold first in a trial, the FPT in the other accumulator is ∞. To this end we adopt the convention that the infimum of an empty set is infinity.

We denote the probability density function (p.d.f.) of T_i by $g_i(t)$ (that is, $g_i(t) = \frac{d}{dt}\Pr(T_i \leq t)$). It is important to note that $\int_0^\infty g_i(t)dt = P_i$, where P_i is the probability of choosing the response i. This probability is not necessarily 1 and so $g_i(t)$ is a *defective* probability density function. A sample path of a Wiener process along with two decision boundaries is shown in the upper panel of Fig. 1.

Another sequential sampling model that we consider in this paper is the *independent race model*. In an independent race model there is a separate information accumulator for each alternative response. Each accumulator is modeled as a stochastic process. These

[1] Even in the Wiener processes considered in Theorems 11 and 12 in Jones and Dzhafarov (2014) the signal to noise ratio should be so large that the information accumulation reduces to a deterministic growth rate (see Smith, Ratcliff, & McKoon, 2014).