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Random sets lotteries and decision theory[☆]

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HIGHLIGHTS

- Is it possible to apply the theory of random sets to decision making under?
- The answer (Proposition 1, Theorem 1, Theorem 2) is yes.
- Our model is compatible with Choquet and MaxMin expected utilities.

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ABSTRACT

We apply random sets theory to decision making under risk. This leads to a unifying concept which is compatible with some types of behavior like the Choquet Expected Utility and MaxMin Expected Utility. We show that the "expected utility" of a random set lottery is easy to calculate. Hence a decision making model with random sets is actually very tractable.

1. Introduction

Suppose that the set over which an agent has to make a choice can expand or contract in a random manner. For instance, if for a given prices' system, the agent's revenue is random then his budget sets (a budget set is a set of goods' vectors that the agent can afford given his revenue and the prices' system) will be random, expanding or contracting according to the value taken by the revenue. How does this agent behave? The idea that a set can be random is captured by the mathematical notion of *random set*. It was mainly used in integral geometry where a random set is considered as a pointed process. But since the eighties it has been used in statistics (see Koshevoy, Mottonen, & Oja, 2003, Molchanov, 2005, Vitale, 1983). In inference statistics, a random set is a confidence region for an estimated parameter. In signal treatment, if you take a grid of pixels, some of these pixels may be randomly colored black and white, and so the resulting picture is a random set (see Goutsias, Mahler, & Nguyen, 1997). Random sets have been used (see Molchanov, 2010 for a review) in econometrics and finance. In econometrics, random sets have paved the way for a thorough body of literature concerning the issue of partially identified models (Beresteanu, Molchanov, & Molinari, 2011, 2012).

We argue in this paper that the concept of random sets can also be useful in decision making theory. This is first because it is a unifying concept which is compatible with several types of behavior: Knight (Bewley, 2011), Savage (1954), Choquet, MaxMin Expected Utility-MEU— (Ghirardato, Maccheroni, & Marinacci, 2004; Gilboa & Schmeidler, 1989). Second we derive (Theorem 1) a property which claims that the "expected utility" of a random set lottery *is reduced to the "utility" of the expectation of a random set*. Hence a decision making with random sets is actually tractable. Third analyzing specifically the issue of the consistency of the ordering of random set lotteries (that is, if a random set lottery is preferred to another random set lottery then the set-valued expectation of the former is preferred to the set-valued expectation of the latter: see

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Eq. (10)), we show (Theorem 2) that an ordering which satisfies consistency conditions and line translation invariance is an ordering of an α-MEU-type decision maker if and only if α is a constant. As a consequence, for the class of symmetric random sets, an α-MEU-type Decision Maker can be viewed as the Choquet expected ordering with symmetric priors.

We study in this paper the risk evaluation of random set lotteries. We consider the Savage framework (Savage, 1954) as a specific case in which the random set lotteries are in the form of constant maps or, equivalently, single-valued random sets lotteries. Specifically, the states space is a single-valued random set within the Savage approach.

Our approach is not new. Ellsberg for instance (Ellsberg, 1961) has criticized the Savage axioms and has considered a lottery with a non-single-valued random set. We recall that in his experiment, there is an urn which contains red, blue and yellow balls such that the red balls constitute one third and there is no prior information about the proportion of the blue balls.

Let us use the Ellsberg paradox in order to illustrate the concept of random set lottery. Let us model Ellsberg's urn as a random set. Let us consider a two-element probability space $\Omega = \{\omega_1, \omega_2\}$ with the Boolean algebra 2^{Ω} and a distribution $P(\omega_1) = \frac{1}{3}, P(\omega_2) = \frac{2}{3}$. Consider the three-dimensional vector space $\mathbb{R}^3$ and denote its basis vectors by $r = (1, 0, 0)$, $b = (0, 1, 0)$ and $y = (0, 0, 1)$. Then a random set R sends ω_1 to the point r and ω_2 to the segment $[b, y] := \{\lambda b + (1-\lambda)y, 0 \leq \lambda \leq 1\}$. Such a random set is not a single-valued random set since, either with probability $\frac{1}{3}$ it might be a vector r, or with probability $\frac{2}{3}$ it might be the segment $[b, y]$. Using the Aumann expectation, the expectation of the random set is the set of expectations of the random vectors $\left(\frac{1}{3}, r, \frac{2}{3}, \lambda b + (1-\lambda)y\right)$, that is the segment $[\frac{1}{3}r + \frac{2}{3}b, \frac{1}{3}r + \frac{2}{3}y]$.

In the Ellsberg paradox, a Decision Maker has to compare four "lotteries": A—to get \$100 if he picks a red ball; B—to get \$100 if he picks a blue ball; C—to get \$100 if he picks a red or yellow ball; and D—to get \$100 if he picks a blue or yellow ball. Experiments show that people prefer A to B rather than B to A, and D to C rather than vice versa. This ordering is inconsistent with the Savage expected utility theory.

In terms of the random set R, the Decision Maker has to compare the following linear functions on $\mathbb{R}^3$, which are specified by the values at vectors r, b and y by:

$$\begin{array}{lll} u_A(r) = \$100, & u_A(b) = \$0, & u_A(y) = \$0; \\ u_B(r) = \$0, & u_B(b) = \$100, & u_B(y) = \$0; \\ u_C(r) = \$100, & u_C(b) = \$0, & u_C(y) = \$100; \\ u_D(r) = \$0, & u_D(b) = \$100, & u_D(y) = \$100. \end{array} \tag{1}$$

Let us calculate the expected utilities. For any affine function $u : \mathbb{R}^n \to \mathbb{R}$, the expected utility for the above random set R is $[\frac{1}{3}u(r) + \frac{2}{3}u(b), \frac{1}{3}u(r) + \frac{2}{3}u(y)]$. Hence the expected utilities are: $E_A = [\frac{1}{3}u_A(r) + \frac{2}{3}u_A(b), \frac{1}{3}u_A(r) + \frac{2}{3}u_A(y)] = \$\frac{100}{3}$; $E_B = [\$0, \$\frac{200}{3}]$; $E_C = [\$\frac{100}{3}, \$100]$ and $E_D = \$\frac{200}{3}$.

As a consequence, choosing between the four "lotteries" A, B, C and D is tantamount to choosing between the four segments $\frac{100}{3}$, $\left[0, \frac{200}{3}\right]$, $\left[\frac{100}{3}, 100\right]$, and $\frac{200}{3}$ (the first and fourth being degenerate segments, singletons).

The problem is that there exists no obvious total preference ordering over a set of segments (Diaye, 1999). Let us consider for instance the natural relation $\preceq_R = \sim_R + \prec_R$ (where + is the union-disjunction operator) defined by $[a, b] \sim_R [c, d]$ if $b \leq c$ and $[a, b] \prec_R [c, d]$ if $b < c$. Then one can observe that E_A and E_B are incomparable with respect to $\preceq_R$, as well as E_B and E_C. Moreover E_C and E_D are incomparable while $E_A \preceq_R E_C$, $E_A \preceq_R E_D$ and $E_B \preceq_R E_D$. Thus even if the Decision Maker chooses according to $\tilde{\preceq}$ a total extension of $\preceq_R$, it may occur that $E_B \tilde{\preceq} E_A$ and $E_C \tilde{\preceq} E_D$. In other words, the Ellsberg paradox is compatible with random sets decision making. Nevertheless *not all behaviors are compatible with random sets decision making* since we allow only weak order (transitive and reflexive) preference over the set of segments (see Section 3).

What is interesting with using random sets lotteries is that we are still in a framework of a *choice under risk*. Of course the lotteries here are no longer necessarily single-valued random sets. The message from our random sets decision making model is that using set-valued lotteries leads us to the conclusion that the Decision Maker's preference on the sets of lotteries is not necessarily a total ordering and this will condition his choice structure. By this move some patterns are feasible which are not allowed in the case of single-valued random sets or the constant utility function.

To conclude, the use of random set lotteries permits us to keep the framework of decision under risk when the way the lottery is modeled is more general, from a single-valued lottery to a set valued one. The purpose of this paper is to present a model of decision making for (non-constant) random sets. We show that such a model includes most models (Etner, Jeleva, & Tallon, 2012) like Choquet or α-MaxMin Expected Utility.

2. Random sets

Let S be a state space and let X be the set of measurable functions on S with respect to the σ-field of measurable sets $\mathcal{S}$. Then the dual space X^* is identified with the set of signed measures on $\mathcal{S}$. We let $\mathcal{F}(X^*)$ denote the set of closed convex subsets of X^*, $\mathcal{K}(X^*)$ denote the set of compact subsets of X^*, and $\mathcal{G}(X^*)$ denote the set of open subsets of X^*, respectively. Let Ω be another state space. Let $\mathcal{A}$ be a σ-algebra over Ω and P be a probability function on $\mathcal{A}$. $(\Omega, \mathcal{A}, P)$ is called a *probability space*.

A mapping $R : \Omega \to \mathcal{F}(X^*)$ is *measurable* if, for any compact set $K \in \mathcal{K}(X^*)$ and any finite collection of open sets $G_i \in \mathcal{G}(X^*)$, $i = 1, \ldots, k$, the set $\{\omega : R(\omega) \cap K = \emptyset, R(\omega) \cap G_1 \neq \emptyset, \ldots, R(\omega) \cap G_k \neq \emptyset\}$ belongs to $\mathcal{A}$. Moreover, the values $P\{\omega : R(\omega) \cap K = \emptyset, R(\omega) \cap G_1 \neq \emptyset, \ldots, R(\omega) \cap G_k \neq \emptyset\}$ form a probability distribution $R_*(P)$ on the σ-algebra $\sigma_{\mathcal{F}}$ of subsets of $\mathcal{F}(X^*)$ spanned by the sets of the form $\{F \in \mathcal{F}(X^*) : F \cap K = \emptyset, F \cap G_1 \neq \emptyset, \ldots, F \cap G_k \neq \emptyset\}$, $K \in \mathcal{K}(X^*)$ and $G_i \in \mathcal{G}(X^*)$, $i = 1, \ldots, k$, $k = 1, \ldots$.

Definition 1. Let $(\Omega, \mathcal{A}, P)$ be a probability space; then the measurable map[1] $R : \Omega \to \mathcal{F}(X^*)$ is said to be a *random set*.

For a finite set S, the space of measurable functions X is the Euclidean space $\mathbb{R}^S$. The dual space of signed measure, X^*, is the set of linear functionals on $\mathbb{R}^S$ and is isomorphic to $\mathbb{R}^S$. The set $\mathcal{F}(\mathbb{R}^S)$ is the set of closed convex subsets of $\mathbb{R}^S$, and the corresponding σ-algebra is the hit-or-miss topology, meaning that in the above-defined σ-algebra $\sigma_{\mathcal{F}}$ we have to consider usual closed, open and compact sets in $\mathbb{R}^S$ (see Matheron, 1975 or Molchanov, 2005).

[1] We note that the literature usually uses the notions of capacity (or hitting) functionals or containment functionals in order to define random sets. A random set R gives rise to the following functions $T_R(K) = P(\{\omega : R(\omega) \cap K \neq \emptyset\})$ and $t_R(K) = P(\{\omega : R(\omega) \subset K\})$, defined for K ranging over closed subsets $\mathcal{F}$ of $\mathbb{R}^n$. These functions are called respectively the capacity functional and the containment functional. However (see Matheron, 1975, Vitale, 1983), setting a random convex compact set R is equivalent to setting a measurable map from a probability space to the set of convex compact of $\mathbb{R}^n$, or else to setting the capacity functional T_R, or to setting the containment functional t_R, or a probability distribution $R_*(P)$ on the σ-algebra $\sigma_{\mathcal{F}}$.