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Structured representations in a quantum probability model of similarity

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HIGHLIGHTS

- We consider a model of basic similarity judgments, based on quantum probability principles.
- We augment this model with Smolensky et al. (2014) ideas for structure in representations.
- We show that this proposal can accommodate the main insights regarding structure in similarity judgments.
- We consider the formal properties of our model and discuss the placement of this work in the similarity, analogy literature.

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ABSTRACT

Recently, Busemeyer et al. (2011) presented a model for how the conjunction fallacy (Tversky & Kahneman, 1983) emerges, based on the principles of quantum probability (QP) theory. Pothos et al. (2013) extended this model to account for the main similarity findings of Tversky (1977), which have served as a golden standard for testing novel theories of similarity. However, Tversky's (1977) empirical findings did not address the now established insight that, in comparing two objects, overlap in matching parts of the objects tends to have a greater impact on their similarity, than overlap in non-matching parts. We show how the QP similarity model can be directly extended to accommodate structure in similarity comparisons. Smolensky's et al.'s (2014) proposal for modeling structure in linguistic representations, with tensor products, can be adapted 'as is' with the QP similarity model. The formal properties of the extended QP similarity model are analyzed, some indicative fits are presented, and, finally, a novel prediction is developed.

1. Introduction

We call quantum probability (QP) theory the rules for assigning probability to events, without any of the physics (Hughes, 1989; Isham, 1989). QP theory is a framework for probabilistic inference alternative to that of classical probability (CP) theory. A case for adopting QP theory, instead of classical probability (CP) theory, in cognitive modeling has been made when human behavior appears at odds with the prescription from CP theory (e.g., Aerts, 2009; Aerts & Gabora, 2005; Blutner, Pothos, & Bruza, 2013; Nelson, Kitto, Galea, McEvoy, & Bruza, 2013; for overviews see Busemeyer & Bruza, 2012; Haven & Khrennikov, 2013; Khrennikov, 2010; Pothos & Busemeyer, 2013; Wang, Busemeyer, Atmanspacher, & Pothos, 2013; Yearsley & Pothos, 2014). Recently, Busemeyer, Pothos, Franco, and Trueblood (2011) presented a model of decision making, based on QP principles, with an emphasis on how the conjunction fallacy (Tversky & Kahneman, 1983), and related findings, emerge. For example, in the conjunction fallacy experiment, participants were told of a hypothetical person, Linda, described very much as a feminist (F) and not at all as a bank teller (BT). Participant responses indicated that $Prob(F \wedge BT) > Prob(BT)$, which is impossible classically. In Busemeyer et al.'s (2011) QP model for this conjunction fallacy, if one assumes that the BT, F possibilities are incompatible, then it can emerge that the quantum probability of $F \wedge$ *then* BT is higher than that of B.

Pothos, Busemeyer, and Trueblood (2013) considered whether the QP decision model could be extended to account for basic similarity judgments. Their motivation was that QP theory is formalized in multidimensional, vector spaces, called Hilbert spaces. The most common, standard way to model basic similarity involves

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multidimensional representations (e.g., Shepard, 1987) and the conceptualization of similarity as a function of distance. For example, such models have been employed in the predominant approaches to categorization (e.g., Goldstone, 1994a; Nosofsky, 1984; Wills & Pothos, 2012). Therefore, since QP representations are also geometric (that is, involve elements in some multidimensional vector space), perhaps QP theory can provide some interesting generalization to the standard distance-based similarity models?

Note first that, by basic similarity judgments, we imply ones that are nonanalytic (in the psychological sense), direct, and immediate. If we accept the view that basic similarity judgments can be modeled as some function of distance, then they have to be consistent with the metric axioms—mathematical requirements that all (simple) functions of distances need to obey. These axioms are intuitively appealing. For example, the symmetry axiom, requires that $distance\,(A, B) = distance(B, A)$, implying that $similarity\,(A, B) = similarity(B, A)$. In one of the most influential studies in the basic similarity literature, Tversky (1977) showed that *all* metric axioms can be violated in similarity judgments of naïve observers. Tversky's (1977) findings have become a golden standard of empirical results that should be accounted for by any basic similarity model and, indeed, have been the focus of theoretical effort in related research ever since (e.g., Ashby & Perrin, 1988; Krumhansl, 1978). It is worth noting that basic, distance-based similarity metrics can be made to violate the metric axioms. For example, symmetry can be violated if $similarity\,(A, B) = p_{AB} \cdot distance\,(A, B)$, where p_{AB} is just a directionality parameter, that is a parameter which can have a different value depending on whether the similarity evaluated is between A and B or between B and A (Nosofsky, 1991). However, the real challenge has been to explore how consistency with Tversky's (1977) findings can emerge from the structure of a basic similarity model.

Pothos et al. (2013) showed how the QP decision model can indeed accommodate Tversky's (1977) key findings, with fairly minor modifications. The objects to be compared in a similarity judgment are represented as subspaces, whose dimensionality depends on the extent of knowledge we have about the objects. Then, similarity judgments are modeled just as conjunctive probabilities of thinking of the first compared object and then the second (one also needs to assume a relevant mental state, that is neutral between the compared objects), that is, $Sim\,(A, B) = Prob(A \wedge then\ B)$ (see also Section 2). For example, Tversky's (1977) famous example of violations of symmetry in similarity judgments was the finding that $Sim\,(Korea, China) > Sim(China, Korea)$, given that participants have more extensive knowledge for China, than Korea (note, actually Red China and North Korea). In the QP model similarity model, this asymmetry can emerge, as long as the dimensionality for the China subspace is greater than that for the Korea subspace.

The application of the QP decision model onto similarity indicates that the formalism can encompass findings from both decision making and basic similarity. Clearly, such a statement needs to be qualified, since the decision QP model addresses only certain kinds of decision making results and, likewise, the similarity one, only certain kinds of basic similarity results. Nevertheless, Pothos et al.'s theory (2013) is encouraging and fits with an overall prerogative to explain as wide a range of empirical findings as possible, with as few explanatory principles as possible. For example, supporting the same QP model in both decision making and basic similarity makes it plausible that the same principles underlie both kinds of cognitive processes.

While our focus until now has been on basic similarly, a somewhat more recent line of work concerns analogical similarity, which partly concerns the study of analogy formation. Simplifying, analogy formation is about how a naïve observer can establish associations between the elements of two representations (e.g., the atom and the solar system). A key focus for models of analogical similarity has been the correspondence between the constituent elements of two compared objects; how do they develop and what is their role in the overall similarity judgment, between the compared objects (Gentner, 1983; Goldstone, 1994b; Goldstone & Son, 2005; Larkey & Markman, 2005; Taylor & Hummel, 2009)? For example, suppose we are comparing two persons, Sue and Linda, with black hair. Surely, this fact would contribute more to the overall similarity judgment between Sue and Linda, compared to an alternative situation, where Sue has black hair and Linda has black shoes. That is, there is intuition and supporting evidence that human similarity judgments are sensitive to the structure of the compared objects. An influential idea in modeling structure in similarity judgments is that feature matches can be aligned or not aligned (Goldstone, 1994b; cf. Markman & Gentner, 1993). That is, parts of one object can be placed in correspondence with the parts of another or not. The implication is that matching aligned parts have a greater impact on similarity judgments than matching unaligned parts, but the latter can increase similarity too.

We note that the distinction we make between basic and analogical similarity is partly one of convenience, as it allows us to easily refer to models of similarity not emphasizing structure (e.g., Ashby & Perrin, 1988; Krumhansl, 1978) and ones that do (Gentner, 1983; Goldstone, 1994b). Cognitively, it is possible that there are differences between judgments of basic similarity and analogy formation (e.g., the latter has been claimed to be sometimes analytic, Casale, Roeder, & Ashby, 2012), but these issues do not concern us presently and the distinction between basic and analogical similarity we employ refers to the objectives and scope of corresponding models.

The purpose of this work is to examine whether the QP basic similarity model can be further extended to cover some key requirements for analogical similarity, notably the way similarity computations are affected by correspondences between representation parts (we do not consider the mechanisms that lead to the discovery of which features align or not; this is an important aspect of research in analogical similarity, but beyond the scope of this work). An extension of this sort cannot be expected to perform as well on analogical similarity results, as thoroughbred models of analogical similarity (e.g., Goldstone, 1994b; Hahn, Close, & Graf, 2009; Larkey & Markman, 2005). Nevertheless, attempting the extension is important: if successful, it will show that the mathematical mechanisms for basic similarity judgments are (plausibly and to some extent) the same as the ones for analogical similarity judgments. Equally, if the general model leads to inferior fits in the novel domain, perhaps a restricted scope is more appropriate. This question of possible equivalence of mathematical mechanisms is separate from the one concerning brain systems (cf. Casale et al., 2012). Note also that most basic similarity models cannot account for structure in similarity judgments (Ashby & Perrin, 1988; Krumhansl, 1978; Tversky, 1977; we return to this point later). Likewise, current attempts to extend analogical similarity models to cover Tversky's (1977) results have difficulties. This separatedness of the literatures on basic similarity and analogical similarity further motivate the present effort to develop a QP model of basic/analogical similarity.

As it turns out, there is a very straightforward way to extend the QP model of basic similarity into one of analogical similarity, using Smolensky, Goldrick, and Mathis's (2014) and Smolensky (1990) ideas for modeling structure in linguistic representations. They were interested in the similarity between linguistic representations, where the role in one representation was compared to the role in the other, and likewise for the fillers (e.g., in relation to phonology, a role could be 'syllable-onset' and a corresponding filler could be 'r', for the word 'rat', as in Smolensky et al.'s, 2014, example in p. 5). They derived their similarity method by employing a tensor product representation, which effectively separated