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Complexity measures and concept learning☆

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HIGHLIGHTS

- We present a new metric to predict the difficulty of supervised concept acquisition.
- Existing mathematical metrics of concept learning explain one SHJ difficulty order.
- That order is for adult human learners facing separable dimension examples.
- This new mathematical metric also explains the order that emerges in all other cases.
- The metric measures information complexity instead of logical complexity.

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ABSTRACT

The nature of concept learning is a core question in cognitive science. Theories must account for the relative difficulty of acquiring different concepts by supervised learners. For a canonical set of six category types, two distinct orderings of classification difficulty have been found. One ordering, which we call paradigm-specific, occurs when adult human learners classify objects with easily distinguishable characteristics such as size, shape, and shading. The general order occurs in all other known cases: when adult humans classify objects with characteristics that are not readily distinguished (e.g., brightness, saturation, hue); for children and monkeys; and when categorization difficulty is extrapolated from errors in identification learning. The paradigm-specific order was found to be predictable mathematically by measuring the logical complexity of tasks, i.e., how concisely the solution can be represented by logical rules.

However, logical complexity explains only the paradigm-specific order but not the general order. Here we propose a new difficulty measurement, information complexity, that calculates the amount of uncertainty remaining when a subset of the dimensions are specified. This measurement is based on Shannon entropy. We show that, when the metric extracts minimal uncertainties, this new measurement predicts the paradigm-specific order for the canonical six category types, and when the metric extracts average uncertainties, this new measurement predicts the general order. Moreover, for learning category types beyond the canonical six, we find that the minimal-uncertainty formulation correctly predicts the paradigm-specific order as well or better than existing metrics (Boolean complexity and GIST) in most cases.

1. Introduction

In a canonical classification learning experiment, human learners are tested on the six possible categorizations that assign eight examples (all possibilities of three binary-valued dimensions) to two equal-sized classes (Shepard, Hovland, & Jenkins, 1961). These classification problems, commonly referred to as the SHJ types, have been instrumental in the development and evaluation of theories and models of category learning. Learning is easiest for Type *I* in which the classes can be distinguished using a simple rule on a single dimension—e.g. all large items are category A and all small items are category B. Learning is most difficult for Type *VI* in which the two classes cannot be distinguished according to any set of rules or statistical regularities. The remaining types (*II*–*V*) are

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intermediate in difficulty. (Table 2 provides a complete description of the six mappings.)

These experiments yield a well-known ordering with a particular pattern across the intermediate types: Type *II* (a logical XOR rule on two dimensions) is learned faster than Types *III–V*, which are learned at the same speed. An update to this traditional SHJ ordering based on a review of the existing literature and a series of new experiments reveals that Type *II* does not differ from Types *III–V* except under particular instructional conditions that encourage rule formation or attention to particular dimensions (Kurtz, Levering, Romero, Stanton, & Morris, 2013).

While this ordering (with or without the recent update) is generally what researchers associate with the SHJ types, there also exists a set of results across a wide variety of learning circumstances in which an entirely different ordering occurs. Specifically, the intermediate types separate into an ordering as follows: $I < IV < III < V < II < VI$. Of particular note is the difficulty in learning Type *II* (along with the non-equivalence of Types *III–V*). There are four separate cases that yield results consistent with this ordering: first, stimulus generalization theory, which generates a prediction of the ordering of the classification problems based on the frequency of mistakes (pairwise confusions) in learning unique labels (i.e., identification learning) for each item (Shepard et al., 1961); second, stimuli comprised of integral dimensions (Garner, 1974) that are difficult for the learner to perceptually analyze and distinguish, such as brightness, hue, and saturation (Nosofsky & Palmeri, 1996); third, learning by monkeys (Smith, Minda, & Washburn, 2004); fourth, learning by children (Minda, Desroches, & Church, 2008).

Since this less well-known ordering occurs across such far-reaching circumstances, we will refer to it as the *general order*; and since the well-known SHJ ordering is only found in one specific learning setting (adult humans learning to classify separable stimuli), we will refer to it as the *paradigm-specific order*. We acknowledge that for some readers, it may seem counterintuitive to dissociate the ordering they are most familiar with from the ordering we designate as general, but in fact it makes good sense to do so.

To provide further details about the evidence for the general ordering, it has been shown that the results for learning the SHJ types with integral-dimension stimuli fully match the general order, i.e. $I < IV < III < V < II < VI$ (Nosofsky & Palmeri, 1996). Since this also corresponds to stimuli generalization theory, these results are interpreted as reinforcing Shepard et al.'s (1961) view that stimuli generalization theory predicts ease of learning unless a process of attention or abstraction can be applied by the learner.

The settings with non-adult or non-human learners match an important characteristic of the general order, that *II* is found to be more difficult than Types *III–V*, while there is only some support for the $IV < III < V$ ordering. In the cross-species research (Smith et al., 2004), four rhesus monkeys were tested on a modified version of the SHJ six types. The core finding is that Type *II* was more difficult for the monkeys to learn than Types *III–V* (which the authors elect to average across in their reporting). In the developmental work (Minda et al., 2008), the researchers modified the SHJ task to be age-appropriate for children of ages 3, 5, and 8. Only Types *I–IV* were tested: Type *II* was the most difficult to learn (consistent with the general rather than the paradigm-specific order). No significant difference between Types *III* and *IV* was observed, however it appears that the researchers did not evaluate the interaction between age of children and their performance on Types *III* and *IV*. From the mean accuracy data, it can be seen that the children show increasingly good performance on Type *III* with age and increasingly poor performance with age on Type *IV*. While we do not have access to statistical support, the available evidence is consistent with the younger children learning Type *IV* more easily than Type *III* (as in the general ordering).

There are two general classes of explanation in the psychological literature on category learning that have been successfully applied to the SHJ types. Mechanistic models, which are implemented in computational simulations of trial-by-trial learning, have been used to explain the paradigm-specific order (i.e. Kurtz, 2007; Love, Medin, & Gureckis, 2004) and some have been shown to account for both the paradigm-specific and general orders (Kruschke, 1992; Nosofsky & Palmeri, 1996; Pape & Kurtz, 2013). The other approach is based on the use of formal metrics to measure mathematical (logical) complexity (Feldman, 2000, 2006; Goodman, Tenenbaum, Feldman, & Griffiths, 2008; Goodwin & Johnson-Laird, 2011; Lafond, Lacouture, & Mineau, 2007; Vigo, 2006, 2009, 2013). These models heretofore account only for the paradigm-specific order.

We put forth a mathematical complexity metric, information complexity, which can account for, on one hand, the paradigm-specific order and, on the other hand, the general order, with a single change in the formula from a *min* to a *mean* operator. Our metric calculates the Shannon entropy (Shannon, 1948) in a classification problem when a subset of the dimensions are specified. The *min* operator identifies the subsets of dimensions which provide the most information (and thus leave the *minimal* uncertainty): this applies to the paradigm-specific order, in which sophisticated learners can observe separable dimensions and may employ abstraction or attention with regard to these dimensions. On the other hand, the *mean* operator averages over subsets of dimensions, and, correspondingly, it applies to the general order, in which learners are less sophisticated or unable to separate dimensions. The logic of this correspondence is described in greater detail in Section 2 (Theory). Among complexity accounts of learning behavior, this new measurement has the advantage of being an analytical function exclusively of observable parameters, i.e. it does not require a heuristic to calculate (Feldman, 2000) nor does it require the fitting parameters to data (Vigo, 2013).

In Section 2, we describe the background of information theory and define the metric. In Section 3, we evaluate the metric's prediction of learning behavior. In Section 3.1, we demonstrate the metric's ability to predict the paradigm-specific and general orders of the SHJ tasks, as well as show it successfully predicts quantitative error rates. In Section 3.2, we demonstrate the metric's ability to predict the paradigm-specific ordering on classification learning tasks beyond SHJ as well or better than the existing metrics (Boolean complexity and GIST) in all cases but one. We also show it successfully predicts the quantitative error rates. The general order setting has not been tested beyond SHJ: this section also, therefore, provides predictions for those future experiments.

2. Theory

In this section, we first provide a comparison of the existing metrics in the literature, which rely on logical complexity, and Shannon entropy, which provides the foundation of our metric. Then, we formally introduce our metric and explain its components.

2.1. Logical complexity versus information complexity

Logical complexity characterizes the length of the shortest description of a system. In an SHJ-style classification, the 'system' in question is a particular categorization. Feldman's Boolean complexity (Feldman, 2000) is a type of logical complexity metric, but there are others, such as Kolmogorov (algorithmic) complexity, which is the length of the shortest program to produce a certain output (Li & Vitâanyi, 2008). These are all related in the sense that they are attempting to construct a minimal set of logical rules that describe a system or process or categorization. The measurement of Boolean complexity begins with the 'disjunctive normal form'