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Inverse, composition, and identity: The case of function and linear transformation

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ABSTRACT

In this report we examine linear algebra students' reasoning about composing a function or linear transformation with its inverse. In the course of analyzing data from semi-structured clinical interviews with 10 undergraduate students in a linear algebra class, we were surprised to find that all the students said the result of composition of a function and its inverse should be 1. We examined how students attempted to reconcile their initial incorrect predictions, and found that students who succeeded in this reconciliation used what we refer to as "do-nothing function" and "net do-nothing function" reasoning. We provide examples of these patterns of reasoning, and propose explanations for why this reasoning was helpful. We also discuss possible sources for this incorrect prediction, and provide implications for classroom practice.

1. Introduction

The concept of function is central to much of secondary and undergraduate mathematics, and there is a robust body of literature examining the nature of students' conception of function (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Confrey & Smith, 1995; Dubinsky & McDonald, 2001; Harel & Dubinsky, 1992; Monk, 1992; Sfard, 1991, 1992). One important context where functions appear is in linear algebra: linear transformations are functions from one vector space to another, often $\mathbf{R}^n$ to $\mathbf{R}^m$, with particular linearity properties (they preserve addition and scalar multiplication). However, the literature on student understanding of linear transformations is relatively sparse, and focuses largely on student difficulties with linear algebra without directly examining function conceptions per se (e.g., Dorier, Robert, Robinet, & Rogalski, 2000; Dreyfus, Hillel, & Sierpinska, 1998; Hillel, 2000; Portnoy, Grundmeier, & Graham, 2006; Sierpinska, 2000).

Moreover, little attention has been paid in the literature to the extent to which students make connections between function in algebraic contexts and transformation in the context of linear algebra (though for recent work in this area, see Zandieh, Ellis, & Rasmussen, 2013). This is of particular interest because prospective high school teachers typically take linear algebra, but how their studies affect their understanding of function is unclear. For example, perhaps their study of transformation in linear algebra actually has a negative effect on their understanding of function. On the other hand, perhaps their study of transformation in linear algebra reinforces and enriches their understanding of function. In either case, their study of linear algebra carries important consequences for their future teaching of secondary school students.

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In order to explore this relationship, we conducted interviews with undergraduate linear algebra students on the extent to which they do or do not construe similarity between function and transformation. These interviews covered a wide range of specific ideas, such as one-to-one, onto, composition, and invertibility. When we conducted and analyzed these interviews, we were surprised to find that all the students made an incorrect prediction about the result of composition of a function with its inverse. In this paper we unpack this surprising result and examine the reasoning of the students who were and were not able to reconcile their incorrect prediction.

2. Theoretical background

Many researchers (e.g., Dubinsky, 1991; Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Sfard, 1991, 1992; Zandieh, 2000) have discussed the dual nature of the function concept. Sfard (1991), for example, asserts that many abstract mathematical concepts, function among them, can be understood either *operationally*, as processes, or *structurally*, as objects. The operational conception is couched in the language of “processes, algorithms and actions,” whereas the structural conception speaks of “static, integrative” objects (p. 4). These two distinct yet complementary aspects of a concept are related reflexively: every process needs objects to operate upon, and processes can come to be understood as objects that can then be acted upon by other processes.

In the framework of Sfard (1991), the development of a concept typically proceeds from operational to structural, passing through three stages called *interiorization*, *condensation*, and *reification*. First, during the *interiorization* stage, the student explicitly performs a process on objects that are already familiar; for instance, students learning about functions may compute tables of functional values by explicitly evaluating functional expressions at particular numbers. Next, in the phase of *condensation*, the student gradually increases in the ability to reason about the process as a coherent whole. In a sense, the procedure becomes a “black box” that objects can be pushed through without attention to the internal workings. Finally, and usually quite suddenly, the concept undergoes a *reification* and becomes an object in its own right, able to be operated upon by other processes.

Another account of the development of the function concept is given by action, process, object, and schema (APOS) theory (Asiala et al., 1996; Breidenbach et al., 1992; Dubinsky & Harel, 1992; for a more thorough treatment of the relationship between APOS theory and Sfard’s account, we refer the reader to Zandieh, 2000). In this framework, the action and process stages are similar to Sfard’s description of operational conceptions of function: students with an *action* view of function operate with functions by simply carrying out calculations on specific numbers, or interpreting the graph of a function as simply a curve or a fixed object in the plane; an underlying interpretation of function as a relationship between two sets is absent. Students exhibiting a *process* view of function are able to think of a function as receiving inputs, performing operations thereon, and returning outputs. With a process conception of function, students can chain two processes together to reason about their composition, or reverse a process to reason about its inverse.

Common student difficulties with composition and inverting are often linked to students’ inability to go beyond an action conception of function (Dubinsky & Harel, 1992). Even (1990, 1993) found that prospective high school teachers without a modern view of function often do not view the result of composition of functions as a function itself; they thus lack understanding of the particular strength of the function concept. While several studies have examined students’ understanding of composition and inverse functions, including the development of items to assess students’ ability to compose a function with the inverse of another function (Carlson, Oehrtman, & Engelke, 2010), little attention has been paid in the literature to the specific case of composition of a function with its inverse.

The development of the function concept from process to object is not without its difficulties. Sfard (1992) notes that many students develop the “semantically debased conception” she refers to as *pseudostructural* (p. 75). Students exhibiting a pseudostructural conception may, for instance, regard an algebraic formula as a thing in itself divorced from any underlying meaning, or a graph as detached from its algebraic representation or the function it represents. Zandieh (2000) describes a pseudostructural conception as a gestalt: that is, “a whole without parts, a single entity without any underlying structure” (p. 108). In the language of Dubinsky, a pseudostructural conception of function is an object view that cannot be “de-encapsulated,” or unpacked to get at the underlying process from which it arose.

3. Methods and background

The subjects of this study were undergraduate students from a linear algebra course at a large public university in the southwestern United States. This course is a sophomore-level course with Calculus I as a prerequisite. It is taken by students in a wide variety of majors, including mathematics, economics, engineering, and computer science. It covered a fairly standard set of topics, including solving systems of linear equations, linear transformations, vector spaces and more abstract linear spaces, determinants, orthogonality, and basic eigentheory. This course was taught by an instructor who was very familiar with the literature on student thinking in linear algebra. The instructor was a member of the research team, but is not one of the authors of this paper.

Several days after the class’s final exam, 10 student volunteers participated in semi-structured hour-long clinical interviews (Ginsburg, 1997) examining their reasoning about the similarities and differences between function and linear transformation. These volunteers were reasonably representative of the students in the course; their grades ranged from A to D+. Interviewer 1 and Interviewer 2 were the third and second authors of this paper, respectively. All interviews