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Explication as a lens for the formalization of mathematical theory through guided reinvention

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ABSTRACT

Realistic Mathematics Education supports students' formalization of their mathematical activity through guided reinvention. To operationalize "formalization" in a proof-oriented instructional context, I adapt Sjogren's (2010) claim that formal proof explicates (Carnap, 1950) informal proof. Explication means replacing unscientific or informal concepts with scientific ones. I use Carnap's criteria for successful explication – similarity, exactness, and fruitfulness – to demonstrate how the elements of mathematical theory – definitions, axioms, theorems, proofs – can each explicate their less formal correlates. This lens supports an express goal of the instructional project, which is to help students coordinate semantic (informal) and syntactic (formal) mathematical activity. I demonstrate the analytical value of the explication lens by applying it to examples of students' mathematical activity drawn from a design experiment in undergraduate, neutral axiomatic geometry. I analyze the chains of meanings (Thompson, 2013) that emerged when formal elements were presented readymade alongside those emerging from guided reinvention.

1. Introduction

Terms like "formal" and "rigorous" are ever-present in discussions of proof and proving, but often lack clear definitions or even descriptions in practice. Sjogren (2010) claimed that formal proof could be understood as the *explication* (Carnap, 1950) of informal proof. Carnap defined explication as the process of supplanting unscientific or informal concepts with scientific or formal ones and provided three criteria for assessing whether a proposed formalization could successfully explicate an informal one. While not every mathematics student thinks that advanced mathematical theory expresses, generalizes, or formalizes their prior mathematical understandings, this is an express goal in the Realistic Mathematics Education (RME) tradition (Freudenthal, 1973; Gravemeijer, 1994). Specifically, Gravemeijer's four stages of mathematical activity describe a progression through which students may develop their less formal or situated understandings into more formal mathematical conceptions via abstraction and generalization. These stages are quite general, having been adapted successfully to describe learning as diverse as children's reinvention of numeric operations (Gravemeijer, 1994) and university students' reinvention of definitions in non-Euclidean geometry (Zandieh & Rasmussen, 2010). In each case, instructional designers must populate and elaborate the stages in the local mathematical context. Toward that end in my teaching and curriculum design in undergraduate, neutral axiomatic geometry, I used Carnap's (1950) explication criteria as supplementary heuristics for elaborating hypothetical learning trajectories (Simon, 1995) for guiding reinvention of a body of theory. In this paper,

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I present how these heuristics, in coordination with a range of existing analytical tools, have proven useful for analyzing student reasoning and subsequently formulating instructional activities.

2. Context, goals, and assumptions

This research took place in a course on neutral axiomatic geometry, which followed the text by Blau (2008), which develops 21 axioms from which it can be proven that any plane satisfying all the axioms must behave like the Euclidean, spherical, or hyperbolic plane. While I set forth the explication lens because I anticipate its broader applicability, there are natural constraints and affordances related to the mathematical context of my research and instructional design. I see students' development of a formal conception of geometric plane as comparable in some respects to many other instances of "apprehending structure" (Simpson & Stehlikova, 2006) in the undergraduate proof-oriented curriculum. Indeed, mathematics educators have successfully developed guided reinvention instructional sequences for algebraic groups (Larsen, 2013) and rings (Cook, 2012), vector space (Rasmussen & Blumenfeld, 2007), and limits (Oehrtman, 2009). However, geometry holds a unique position regarding proof-oriented mathematics historically (Freudenthal, 1973) and pedagogically (González & Herbst, 2006) because of the relation between empirical and deductive thinking. Most undergraduate students in the USA were taught some form of Euclidean geometry in high school and all students bring a history of spatial experiences to their geometric learning, which can either support or hinder systematization of geometric knowledge (e.g. Human & Nel, 1984). This context affords using explication to design a guided reinvention instructional program because:

1 students enter undergraduate geometry courses with rich intuitive and informal understandings,
2 the course which housed these investigations defines and formalizes very basic experiential concepts such as distance, spatial arrangement, rays, and lines, and
3 while the modern axiomatic tradition emphasizes translating geometric proofs into some purely syntactic calculus native to the axiomatic system, students need semantic insight (Weber & Alcock, 2009) to guide geometric proof production.[1]

Whereas previous research tends to dichotomize semantic and syntactic proof production (Alcock & Inglis, 2008), by adopting an explication lens I problematize students' ability to coordinate the two.

So, throughout my research I assume the goal that students should conceive the axiomatic body of theory as expressing and formalizing their prior learning and experiences. This is consistent with the RME tradition and the notion of *advancing mathematical activity* (Rasmussen, Zandieh, King, & Teppo, 2005). As Rasmussen et al. (ibid) point out, "This transition [from intuitive concepts to formal concepts] is indeed difficult when students' intuitive basis founded on experience is an island (Kaput, 1994) separated from their reasoning based on formal definitions and logical deductions" (p. 71). However, such an approach to axiomatic instruction is a strict departure from many traditional views thereof. For instance, Kershner and Wilcox (1950) warn readers of their text "Unless *all* suggestions conveyed by [mathematical terms] from past association are persistently ignored, a multiplicity of meanings may arise" (p. 17) clearly suggesting that learning axiomatics should be divorced from prior understandings. A guiding assumption of this design experiment is that guided reinvention can support greater coordination of semantic insight with syntactic tools and precision (Weber & Alcock, 2009). The inherent challenge is to help students conceive exactly which aspects of their informal geometric understanding are explicated, how they can be embedded in a body of theory (e.g. definition, axiom, or theorem), and the mathematical relations between these explications (e.g. defining dependencies, logical dependencies, and proofs). Analytically, I describe such coordination in terms of students' construction and coordination of formal and informal *meanings* (in the sense of Piaget & Garcia, 1991; Thompson, 2013) for key geometric concepts, which I intend for the reinvention process to privilege. That is to say, I desire instructional sequences by which students develop meanings that simultaneously reflect their informal geometric reasoning and that contribute to the production of formal, axiomatic proof. Carnap's (1950) explication criteria provide tools for assessing the viability of student meanings for these dual goals. This paper thus demonstrates how the explication criteria:

1 aided analysis of student meanings incompatible with the learning goals,
2 supported the development of alternative instructional tools that address student difficulties, and
3 characterized examples of student meanings emerging within the reinvention process that afforded the fruitful coordination of semantic and syntactic reasoning in proof production.

3. Theoretical tools

In this section, I lay forth the various components I draw from the literature for characterizing the psychological explication lens for guided reinvention. Thus, Sections 3.1 and 3.2 define explication and exemplify my adaptation of the notion in neutral axiomatic geometry. Section 3.3 outlines Gravemeijer's (1994) stages of formalization in guided reinvention contexts

[1] While they are not identical, I will use Weber and Alcock's (2004, 2009) *semantic* and *syntactic* terminology as a starting point for identifying students' proof production activity as *informal* or *formal*. I concur with Stylianides (2007) that the formality or rigor of an argument is best conceptualized as a gradient, but consistent with his Boolean interpretation of the "proof threshold," I shall maintain the dichotomy between formal and informal.