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The two-change problem and calculus students' thinking about direction and path☆

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ABSTRACT

The purpose of this paper is to propose the *two-change problem* as an important conceptual issue that students experience as they reason about the rate of change of multivariable functions. This paper presents the results of interviews to illustrate how students conceived of the two-change problem and attempted to resolve it. In doing so, they developed initial notions of the dependence of rate of change on direction and path. The paper closes by discussing the implications of the two-change problem for engendering useful ways of thinking about rate of change in three or more dimensions.

1. Introduction

Many mathematical ideas entail reasoning about and representing changes among many variables in a system. This reasoning and representation entails thinking systematically about the rate of change of variables with respect to one another. In the multivariable case, this involves holding variables constant and parameterizing certain variables in a coordinated way. For the two-variable function case, this entails thinking about rate of change in a direction. However, it is unclear how students might come to think about rate of change in these ways. For example, little is known about what understanding(s) of rate of change of a single variable function help create opportunities for students to conceive of the dependency of rate of change on direction and path in three or more dimensions. This paper reports on a study with calculus students that intended to gain insight into the following research question: *what issues might students encounter as they conceive of rate of change of a two-variable function in a direction, and how might their ways of thinking develop to address these issues?* The purpose of this paper is to propose the *two-change problem* as an important conceptual issue that students experience as they reason about rate of change of two-variable functions.

Briefly, the *two-change problem* occurs when students notice there are two (or more) rates of change of a multivariable function "at" a point $(a, b, f(a, b))$, but they also believe that rate of change must be expressed as a single value at that point. For instance, consider one student's computer sketch noting at least two different rates of change (one corresponding to changes in x, the other corresponding to changes in y) while he still believed it was necessary to combine them into a single rate of change (in purple, top right, corresponding to a combination of a change in x and/or change in y) (Fig. 1).

This paper focuses on the emergence of the two-change problem from the analysis of clinical interviews with calculus students and considers the implications of this problem for conceiving of the more general notion of direction and path.

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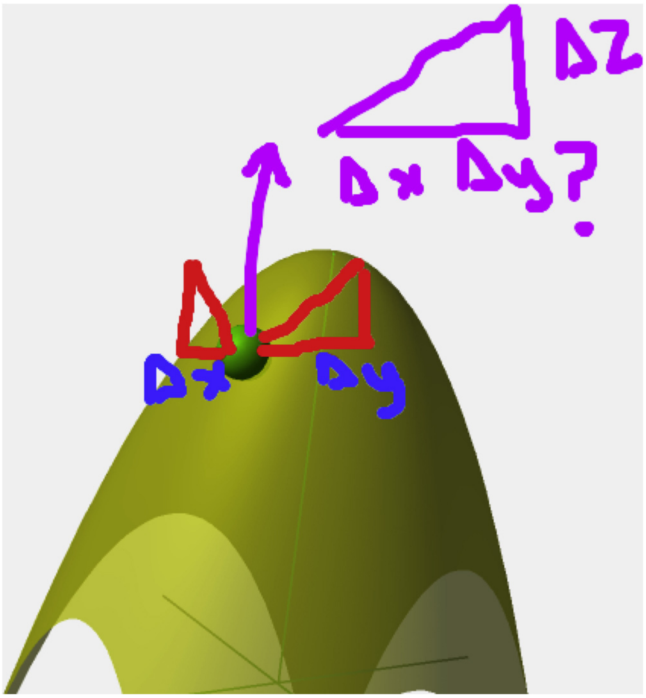

Fig. 1. Student computer sketch work illustrating the two-change problem.

2. Existing literature and important extensions

A number of researchers have identified difficulties students have thinking about rate of change. Examples of these issues include thinking about a graph as representative of its derivative (Nemirovsky & Rubin, 1991), confounding average and instantaneous rate of change (Orton, 1983), conceptualizing rate as the slant of a graph (Weber & Dorko, 2014) and inattention to how fast quantities are changing with respect to one another (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). To expand on this last issue, thinking about rate of change as a measurement of how fast quantities are changing is foundational to calculus, yet many students possess difficulties reasoning about rate in this way (Carlson, Larsen, & Jacobs, 2001; Rasmussen, 2001; Thompson & Silverman, 2008). Why is this? Thompson (1994b) found that the difficulties students displayed in understanding the fundamental theorem of calculus arose from impoverished concepts of rate of change and incoherent images of functional covariation, which requires both an image of how two quantities have changed in tandem and a multiplicative comparison of those changes.

A number of other studies (Moore, Paoletti, & Musgrave, 2013) have supported this claim but all of them have occurred in the context of questions about functions of one variable. Thus, it is not immediately clear how these problems (and sources of them) extend to functions of two or more variables. For instance, Yerushalmy's (1997) work provides an indication of natural questions that might arise as students conceptualize rate in three dimensions. She posed natural questions: such as how to think about dependence in a system with three quantities, how to "combine" two separate relationships for a single phenomenon, and how to represent multiple quantities in a single graph (pp. 437, 441).

These are the types of insights, and questions, that initially generated the research question driving this study: what issues might students encounter as they conceive of rate of change of a two-variable function in a direction, and how might their ways of thinking develop to address these issues? Subsequent sections of this paper explain how the two-change problem emerged as part of the answer to this research question, which in turn generated a number of other questions to consider that were evaluated with a sequence of clinical interviews.

3. Theoretical framework

This study relied on Piaget's structuralism to characterize students' ways of thinking (Piaget, 1971b). Piaget described structures as self-contained entities that interact with a world (internal or external) and are characterized by transformation. The interaction is actually the basis for those transformations. Piaget's (1971a) constructs of *assimilation* and *accommodation* are key elements of the transformation of these structures and were central to the analysis of student thinking in this study. Briefly, this study took assimilation to mean imbuing an object with meaning using an existing cognitive structure (i.e. rate of change as the slope of a line). For instance, suppose a student conceives of a graph as a shape rather than a representation of a