Journal of Mathematical Behavior 42 (2016) 20–32

Contents lists available at ScienceDirect

The Journal of Mathematical Behavior

journal homepage: www.elsevier.com/locate/jmathb

A false belief about fractions – What is its source?

Marianna Ciosek*, Maria Samborska

Institute of Mathematics, Pedagogical University of Cracow, Poland

ARTICLE INFO

Article history:
Received 30 April 2015
Received in revised form
31 December 2015
Accepted 14 February 2016
Available online 27 February 2016

Keywords:
Fractions
Problem solving
False belief
Conditional reasoning

ABSTRACT

This paper presents the results of interviews with 174 participants solving a problem of elementary mathematics, connected with the part–whole approach to fractions. The motive for the investigation was a specific kind of difficulty observed during a case study conducted to verify the elementary school student's understanding of the concept of fractions. The authors decided to examine the problem in a broader population of mathematics learners at different levels of education: from elementary school to university students and graduates of science majors. Approximately 65% of respondents reported the wrong answer immediately after reading the fraction problem taken from the fourth grade of elementary school. Detailed analysis of the respondents' performance showed that the source of many wrong answers was a false belief about fractions: *The only way to get 1/n of a given whole is to divide this whole into n equal parts*, not yet described in educational literature.

> Sometimes one phenomenon, one example, apparently isolated, even seemingly strange, is reflected as a sudden flash in our pedagogical intuition and can turn us towards an essentially important problem.
> A.Z. Krygowska

1. Introduction and background

The article, in fact, provides an analysis of the behavior of people at different levels of mathematical knowledge in their attempts to solve an elementary but not typical problem regarding fractions. Thus it refers to the concept of fractions and to the mathematical problem solving process. This chapter includes a theoretical background of the study for both mentioned issues.

The approaches to teaching fractions in schools have changed over time. Before the 1970s, emphasis was placed on the arithmetic of fractions and this topic was presented to the students in a formal way. Streefland (1990) mentioned Dienes' book "Fractions An Operational Approach", published in 1967, as an example of teaching fractions in a very formal manner. In the 1970s and early 1980s, the stress on computational skills weakened, but developing these skills still dominated. Large-scale investigations regarding students' difficulties in different areas of mathematics, e.g. the CSMS project in England (Hart, 1980; Hart, Brown, Kuchemann, Kerslake, Ruddock, & McCartney, 1981), "Tweede Wiskunde Project" in The Netherlands (Pelgrum, Eggen, & Plomp, 1983), and the research into students' knowledge and skills concerning fractions (e.g., Ekenstam & Greger, 1982; Kerslake, 1986) revealed the complexity of the concept of fraction and the difficulties in its proper understanding. Educators became more and more aware that the topic of fractions is difficult for pupils and more attention should be paid to the concept of a fraction itself. In the theoretical considerations of the last decades, different types of phenomena

* Corresponding author.
E-mail addresses: ciosek.maria@gmail.com (M. Ciosek), maria.samborska@tlen.pl (M. Samborska).

http://dx.doi.org/10.1016/j.jmathb.2016.02.001
0732-3123/

shaping the concept of a fraction and different aspects of this concept were discussed (e.g., Freudenthal, 1983; Turnau, 1990). New concepts for developing the understanding of fractions in school education were presented, e.g., by Kerslake (1986), Streefland (1990), Ciosek, Kubinová, Tichá, and Turnau (2000), Padberg (2000), Tichá (2000), and Steffe and Olive (2010). Nowadays, in school curricula, there is no formal definition of a fraction. Instead, various intuitive aspects of a fraction are exposed: a fraction as a part of a whole, as an operator transforming a quantity, as a ratio, or as a number. This article refers to the part–whole aspect of fractions, studied among others by Kerslake (1986), Streefland (1990), Kieren (1993), Cardoso and Mamede (2010), Steffe and Olive (2010). In our study, we examine the understanding of this aspect through analysing the process of solving a non-standard problem related to the part–whole model of fractions.

Theoretical considerations about mathematical problem solving are multidimensional. One of the dimensions is the modeling of problem solving performance. Polya (1973) created a four-phase model for problem solving that is familiar to every researcher who has undertaken a study of this matter. This model specifies four stages of an individual's work while solving a problem: understanding, planning, carrying out the plan, and looking back. Another dimension of the problem solving research focuses on factors that influence the problem solving performance (e.g., Schoenfeld, 1985, 1992; Lester, 1985; Silver, 1987). Schoenfeld (1985) distinguished four categories of knowledge and behavior necessary for the understanding the mathematical problem solving performance: *resources* (the individual's knowledge brought to bear on the problem), *heuristics* (strategies and techniques for solving non-standard problems), *control* (global decisions regarding the selection and implementation of resources and strategies), and *belief systems* (one's mathematical world view). In our study, special attention is paid to resources – the subjects' knowledge about fractions revealed in the problem solving process. But some resources may be incorrect. A common kind of flawed resources are misconceptions, also called *false beliefs*, *misbeliefs*, *faulty beliefs* (Graeber & Tirosh, 1989) or *false convictions* (Pawlik, 2007). They can be described as the individual's (subjective) personal knowledge which is inconsistent with objective mathematics. According to Pawlik, *false conviction* can be characterizes it in the following way: a solution based upon false conviction is produced as if the solver already knew the answer. He does not check anything and does not search for anything, he simply forms the answer through materializing the first idea that hit his mind. Ciosek and Turnau (2015) add to this characteristic a cognitivist view: "A false conviction (false belief) is a relationship or regularity wrongly taken as generally true. It may be revealed when the solver of a problem refers to it (explicitly or implicitly) while justifying his/her (wrongful) solution." Among the empirical research studies connected with the kind of false belief mentioned above are those focused on arithmetic operations (e.g., Bell, 1988; Fischbein, Deri, Nello, & Marino, 1985; Graeber & Tirosh, 1989; Tsamir & Tirosh, 2002). This research revealed school students' and teachers' beliefs, such as: *multiplication makes bigger*, *division makes smaller, quotient must be less than dividend, perform-the-operation belief* (division by zero results in a number). According to Fischbein (1987), these beliefs are examples of intuitive beliefs defined as immediate forms of cognition and refer to statements and decisions which exceed the observable facts. Characteristics of intuitive beliefs are, among others: self-evidence, intrinsic certainty, extrapolativeness and globality. Fischbein stated that sources of the beliefs mentioned earlier are primitive models of arithmetic operations. In our opinion, the false belief about fractions considered in this study is another example of intuitive belief.

2. Motives and purpose of the investigation

In 2008, a case study was conducted to verify the elementary school student's understanding of the concept of a fraction. In the study, various aspects of fractions were accounted for: a fraction as a measure, as an operator, as a ratio of two quantities, as a quotient of two integers (Ciosek et al., 2000). A student teacher of mathematics (Bisaga, 2008) interviewed fourth grade students, then fifth and sixth graders. Among them was David (grade 4, rated "very good" in math) observed during his individual work on problems involving fractions. One of the problems David was solving (see Fig. 1) was unknown to him as it was taken from a textbook (Ciosek, Legutko, Turnau, & Urbańska, 2005) not used in his classroom. As other students, David received the contents of the problem on a paper sheet, on which he wrote his solution. He was asked to inform the observer of his ideas. Sometimes he was asked additional questions to clarify his written or oral statements. David's comments were recorded on a voice recorder.

Once David got familiar with the task, he thought about it for a moment, then said: *The shaded part is not one third*. Then he began to count boxes in the outer part of the frame (starting from the top row). He stopped short soon and said with firmness in his voice: *The yellow part does not constitute one third of the figure*. He then justified his answer in the following way: *I'm sure it will not be one third, because here* (he pointed to the smallest part of the frame) *there are less boxes than the orange ones, and there are more white ones, those further away from the center, than the orange ones*.

The problem seemed to the trainee teacher interesting enough to be put to her colleague, a second-year student at a technical university. His spontaneous answer was: *In the frame the orange part is not one third of it for "at a glance" you see that these parts are of different sizes*. After a moment of reflection he said: *Something did not fit in my answer, so I counted the boxes of the frame. First I counted the orange ones and arrived at the number 32, then I counted the blank ones and summed them up. I got the number 96. I took the ratio of these two numbers and got 32/96, and it is 1/3. So the orange part of the frame is 1/3 of the entire frame. Well, my intuition failed me, but it's important, though, that I was able to arrive at the correct solution*.

The fact that both the elementary school student and the university student gave a spontaneous response "No" to the situation under consideration provokes reflection. Observations, made in the years 2008–2012, which were carried out several times in groups of trainee mathematics teachers when solving the *Frame Problem* showed that most of the answers were similar to David's. This fact might indicate that the fraction 1/3 is associated, not only to a primary school student, but