Journal of Mathematical Behavior 42 (2016) 92–108

Contents lists available at ScienceDirect

The Journal of Mathematical Behavior

journal homepage: www.elsevier.com/locate/jmathb

"Multiply by adding": Development of logarithmic-exponential covariational reasoning in high school students

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ARTICLE INFO

Article history:
Received 14 July 2015
Received in revised form 29 February 2016
Accepted 11 March 2016
Available online 26 March 2016

Keywords:
Covariational reasoning
Logarithmic-exponential covariation
Teaching experiment

ABSTRACT

This paper reports the development of logarithmic-exponential reasoning in high school students in a teaching experiment as a contribution to the growing body of research addressing covariational reasoning. The design of the teaching experiment articulates the *static perspective of exponential covariation* and a reformulation for the discrete case of the *dynamic perspective of the covariation*. The static perspective considers that amounts of one quantity varying in arithmetic progression are associated or juxtaposed with amounts of another quantity varying in geometric progression. On the other hand, the dynamic perspective coordinates two varying quantities while attending the ways in which they change in relation to each other. The conceptual framework also takes into account the results of previous epistemological-historical studies about the development of the concepts of exponential and logarithmic functions. The data analysis aims to define the logarithmic-exponential covariational reasoning of the participants through the identification of the mental actions and levels of reasoning achieved during the teaching experiment. The results show that the teaching experiment contributed to the students' perception of the changes in the involved variables and to infer the juxtaposition of the operations: "multiplying by adding" and "divide by subtracting". Therefore, we consider that the sketch of a graph and algebraic expressions demonstrate the development of logarithmic-exponential covariational reasoning in students.

1. Introduction

Covariational reasoning is a widely used construct within a growing body of research focused on explaining students' process of creating a relationship between quantities. This entails the mental actions—that is non-visible actions that individuals apply to material or immaterial objects, accessible to third parties through a symbolic medium—involved in conceiving two quantities as varying in tandem (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Saldanha & Thompson, 1998). It has been shown that covariational reasoning is fundamental for students' understanding of numerous secondary and post-secondary mathematics subjects, such as exponential relationships (Castillo-Garsow, 2010; Confrey & Smith, 1995; Ellis, Ozgur, Kulow, Williams, & Amidon, 2012), trigonometry (Moore, 2010), rate of change (Johnson, 2012, 2015a, 2015b), function (Carlson

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http://dx.doi.org/10.1016/j.jmathb.2016.03.003

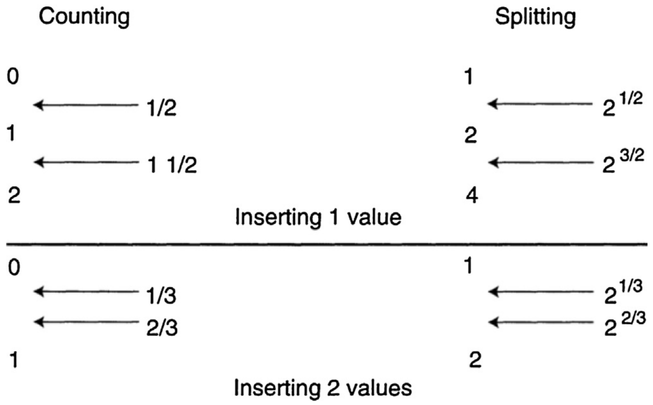

Fig. 1. Interpolation in counting and splitting worlds contrasted (Confrey & Smith, 1995, p. 82).

et al., 2002; Johnson, 2012, 2015a, 2015b), the fundamental theorem of calculus (Thompson, 1994a), graphics of functions (Carlson et al., 2002; Moore, Paoletti, & Musgrave, 2013) and differential equations (Castillo-Garsow, 2010).

The aim of this research is to widen the comprehension of the development of covariational reasoning of exponential and logarithmic functions in high school students. To achieve this, we used two conceptual frameworks of covariational reasoning (Carlson et al., 2002; Confrey & Smith, 1995) to design a teaching experiment that allows the student to connect, within the framework of discrete variation, dynamic and static covariational reasoning.

1.1. Covariational reasoning in function concept

There is a growing interest in the specific research area of covariational reasoning in the context of students' concept of function (Carlson et al., 2002; Falcade, Laborde, & Mariotti, 2007; Hitt, & González-Martín, 2015; Hoffkamp, 2011; Johnson, 2012, 2015b; Oehrtman, Carlson, & Thompson, 2008; Saldanha & Thompson, 1998). Some authors point out that covariational reasoning can be developed by modelling dynamic events (Carlson et al., 2002; Hitt, & González-Martín, 2015; Johnson, 2012, 2015b; Oehrtman et al., 2008; Saldanha & Thompson, 1998) and others prefer dynamic geometry software (Falcade et al., 2007; Hoffkamp, 2011). Altogether, these researchers show the fundamental role of covariational reasoning (along with quantitative reasoning) in the learning process of the notion of function from a dynamic perspective and they all recommend, in accord with Oehrtman et al. (2008), that school curricula and instruction "include a greater focus on understanding ideas of covariation and multiple representations of covariation (e.g., using different coordinate systems), and that more opportunities be provided for students to experience diverse function types emphasizing multiple representations of the same functions" (p. 32).

1.2. Covariational reasoning in particular functions and exponential functions

Specific functions have received special attention in covariational reasoning studies from a dynamic perspective: lineal and quadratic functions (Ellis, 2011a, 2011b), trigonometric functions (Moore, 2010, 2012, 2014) and exponential functions (Castillo-Garsow, 2010; Confrey & Smith, 1994, 1995; Ellis et al., 2012). The research about trigonometric functions shows, for example, that constructing a system of meanings for angle measure, quantity, measurement and covariation, in both circle and triangle contexts, is necessary for understanding trigonometric functions (Moore, 2014) and that "quantification of angle measure [is] a critical springboard for a covariational approach to the sine function" (p. 134).

Conceptually, there are two different covariational approaches to the exponential function. The first one, as explained by Confrey and Smith (1995), argues that the covariational approximation of a function is produced when it is seen as a juxtaposition of two sequences, each one constructed independently through data patterns. There are two different and simultaneous variations that affect each other mutually. Particularly, exponential covariation is the coexistence of variations in constant rates and in constant differences or, in other words, juxtaposing the amounts of one quantity varying in arithmetic progression with amounts of another quantity varying in geometric progression. Confrey and Smith suggest the emergence of exponential functions through numerical tabulation arrangements. These data grids promote the construction of a "counting world" and a "splitting world" and their juxtaposition.

Confrey and Smith propose a distinction between two classes of "number worlds". An additive or counting world is based on an arithmetic sequence, where successive elements have a constant difference, and a multiplicative or splitting world is based on a geometric sequence, where successive elements have a constant multiple. Fig. 1 shows the juxtaposition between the counting world and the splitting world. Thus "the construction of a counting and a splitting world and their juxtaposition through covariation provide the basis for the construction of an exponential function" (Confrey & Smith, 1995, p. 80). In a