Journal of Mathematical Behavior 43 (2016) 1–19

Contents lists available at ScienceDirect

The Journal of Mathematical Behavior

journal homepage: www.elsevier.com/locate/jmathb

The tacitly nested number sequence in sixth grade: The case of Adam

Catherine Ulrich

School of Education, Virginia Tech, 306 War Memorial Hall, Blacksburg, VA 24061, United States

ARTICLE INFO

Article history:
Received 9 September 2015
Received in revised form 26 March 2016
Accepted 16 April 2016
Available online 21 May 2016

Keywords:
Constructivist teaching experiment
Middle grades mathematics
Number sequence

ABSTRACT

Student learning in the middle grades is afforded and constrained by basic numerical operations characterized in the number sequences framework (Steffe, 2010a), originally developed with younger children. I investigated its use with four sixth-grade students through a constructivist teaching experiment. I found that one student, Adam, was able to strategically operate on composite units, which is a strong indication that the he had constructed an explicitly nested number sequence (ENS). However, he was not able to reversibly disembed subsequences or construct iterable units of 1, both principal operations of an ENS. I conclude that Adam was constrained to the construction of the less sophisticated tacitly nested number sequence (TNS), and that his advanced behavior is a natural outgrowth of his increased experience operating with a TNS. I suggest a refinement of the number sequences framework to support research and teaching at the middle grades.

1. Introduction

In the middle school years, there are many new mathematical topics students grapple with—fraction multiplication and division, arithmetic with negative numbers, and algebraic notation—all of which are notoriously difficult for students to learn. We know from extant and ongoing research (e.g., Steffe & Olive, 2010; for fractions; Ulrich, 2012, for integers; and Hackenberg & Lee, 2015 for algebraic notation) that student operations in these domains are afforded and constrained by their basic number operations. Therefore, it is important for middle grades mathematics educators and researchers to have a firm grasp on the number operations that middle grades students have available to adapt and build on in making sense of these new domains.

Steffe and colleagues (Olive, 2001; Steffe & Cobb, 1988; Steffe, 2010a; Steffe, Glasersfeld, Richards, & Cobb, 1983; Ulrich, 2015, 2016) developed four empirically-based stages to characterize student development of arithmetic meanings and strategies using their counting number sequences. These number sequences were developed over the course of multiple teaching experiments with students ranging from first grade through fifth grade. In the course of a teaching experiment with sixth-grade students, I serendipitously came across a student, Adam,[1] who gave strong indications of being at the second stage—limited to construction of a tacitly nested number sequence (TNS)—but who also gave strong indications of being at the third stage—limited to the construction of an explicitly nested number sequence (ENS). The case of Adam forced me to

E-mail address: culrich@vt.edu

[1] All participants' names are pseudonyms.

http://dx.doi.org/10.1016/j.jmathb.2016.04.003

reexamine fundamental assumptions about the mathematical abilities of students at various levels of the number sequences, especially when they have the increased mathematical experiences and problem solving tools of a middle grades student.

Just as extreme examples in mathematics are often used to clarify the boundaries of definitions, I will use a case study of Adam to refine the theory of the tacitly nested and explicitly nested number sequences by isolating out the most important characteristics of an explicitly nested number sequence: the construction of disembedding[2] and an iterable unit of 1. By the end of my teaching experiment, I could confidently categorize Adam as constrained to the construction of a TNS. In particular, I show how a TNS student could operate with composite units in surprisingly sophisticated ways without the efficiencies in numerical reasoning made possible by a reorganization of a TNS into an ENS. I argue that Adam's seemingly paradoxical mathematical behavior was due to his extreme fluency with the operations of his TNS. This fluency implies that he has been constrained to the construction of a TNS for a significant amount of time, which would be entirely plausible given that Steffe and colleagues document the construction of a TNS among some second grade students (e.g., Steffe & Cobb, 1988).

2. Theoretical framework

This research was carried out from a radical constructivist paradigm (Glasersfeld, 1995), which means that I assume that each of my participants has constructed a way of operating that is viable in their mathematical realities. Because I have no direct access to their knowledge, I make conjectures about the nature of their mathematical realities based on their observed ways of operating. In doing so, I form explanatory models for their mathematical ways of operating. In forming these models, I readily utilize constructs from other researchers, where appropriate, several of which I will clarify here.

In order to talk about the mathematical knowledge of a student, I will utilize the concept of a *scheme* (see Glasersfeld, 1979; Piaget, 2001/1977). A scheme consists of an assimilating structure, the activity or procedure of the student, and the result of the activity. For example, a plurality of countable objects might be the necessary feature in a child's experience to allow the child to initiate his or her counting scheme. In that the countable items for a student change over time, the assimilating structure for the student's counting scheme is changing as well. *Assimilating structures* are, more generally, ways of organizing experiences that are open to a student. For example, once a scheme has become assimilatory to a child, the child no longer has to carry out the action of the scheme before considering further operation on the results of the scheme. When I talk about the mathematics of a student as being more *sophisticated* than the mathematics of another student, I mean that their assimilating structures are more elaborated and perhaps are at a higher level of abstraction, allowing the construction of more powerful schemes.

2.1. Number sequence overview

Steffe developed four stages of numerical development, which can be characterized by the increasingly complex numerical structures students can construct in activity and as assimilatory constructs (Ulrich, 2015, 2016). Steffe calls the assimilatory constructs for counting behavior *number sequences.* They proceed in hierarchic fashion from an initial number sequence (INS) to TNS to ENS to a generalized number sequence (GNS). An INS is the first truly numerical counting structure in which students can count without perceptual items available. Each number word can be used to stand in for the actual counting activity from 1 up to that number word, allowing the student to count on from numbers other than 1. The primary development leading to a TNS is the construction of composite units, in which a unit larger than 1 can be used to segment the number sequence and solve problems. An ENS contains reprocessed units, known as *iterable* units, which allow for a more explicit awareness of and operation on nested subsequences. A GNS generalizes the use of iterable units to composite units, allowing for increasingly complex multiplicative structures.

In this article, we consider in depth the two middle stages: a tacitly nested number sequence (TNS) and an explicitly nested number (ENS). Two of the students, Justin and Lily, whose mathematical behavior I use as foils to compare to Adam's mathematical behavior, had already constructed a GNS. However, this analysis will focus on the distinctions between a TNS and ENS, both of which are discussed in more detail below.

2.1.1. Tacitly nested number sequence

A TNS results from a general reorganization of the INS triggered by the construction of composite units–the result of unitizing a subsequence to form a unit larger than 1 that can be used to segment the number sequence. For example, a TNS[3] student might answer a question such as "How many threes are in twelve?" by using a composite unit of 3 to segment their counting sequence to twelve and keeping track, implicitly, of how many times a group of three number words are uttered: *one two THREE [puts up a finger] four five SIX [puts up a finger] seven eight NINE [puts up a finger] ten eleven TWELVE [puts up a finger]. . .4 times*. An INS student would have to use additional supports, probably perceptual items, to make and count the groups of 3.

[2] *Disembedding* will be used to refer to a reversible disembedding operation, which is distinct from attentional bounding (cf. *disembedding* in Steffe, 1988, 1992).

[3] TNS, ENS, and GNS will be used as adjectives to indicate a student who has constructed a number sequence and is constrained by it.