Journal of Mathematical Behavior 43 (2016) 20–34

Contents lists available at ScienceDirect

The Journal of Mathematical Behavior

journal homepage: www.elsevier.com/locate/jmathb

Conditions for proving by mathematical induction to be explanatory

Gabriel J. Stylianides[a,*], James Sandefur[b], Anne Watson[c]

[a] University of Oxford, Department of Education, 15 Norham Gardens, Oxford, OX2 6PY, UK
[b] Georgetown University, Washington DC, USA
[c] University of Oxford, Oxford, UK

ARTICLE INFO

Article history:
Received 18 January 2015
Received in revised form 27 March 2016
Accepted 16 April 2016
Available online 24 May 2016

Keywords:
College/university mathematics
Examples
Explanation
Proof by mathematical induction
Problem design
Proving

ABSTRACT

In this paper we consider *proving* to be the activity in search for a proof, whereby *proof* is the final product of this activity that meets certain criteria. Although there has been considerable research attention on the functions of proof (e.g., explanation), there has been less explicit attention in the literature on those same functions arising in the proving process. Our aim is to identify conditions for proving by mathematical induction to be explanatory for the prover. To identify such conditions, we analyze videos of undergraduate mathematics students working on specially designed problems. Specifically, we examine the role played by: the problem formulation, students' experience with the utility of examples in proving, and students' ability to recognize and apply mathematical induction as an appropriate method in their explorations. We conclude that particular combinations of these aspects make it more likely that proving by induction will be explanatory for the prover.

1. Introduction

Although there has been considerable research attention on the functions of proof (explanation, verification, generation of new knowledge, etc.), there has been less explicit attention in the literature (to the best of our knowledge) on the same functions arising in the proving process. In relation to proof, there are also different perspectives in the literature about the various functions proof can serve. One perspective is to consider these functions as the purposes proof can serve for the prover or the reader of the proof (we call this the *subjective perspective*), whereas an alternative perspective is to consider the functions as characteristics of the text of the proof (we call this the *absolutist perspective*). There are also hybrid perspectives that combine aspects of these two perspectives.

The terms 'proof' and 'proving' have also been used in a number of different ways in the literature. It is beyond the scope of this paper to discuss the different meanings of 'proof' and 'proving' (see Stylianides, Stylianides, & Weber, 2016 for a review). It is important, though, to clarify our use of these terms herein. We consider *proving* to be the activity in search for a proof (e.g., Stylianides, 2007; p. 290), whereby *proof* is the final product of the proving activity that meets the following criteria: it is "a valid argument based on accepted truths for or against a mathematical claim that makes explicit reference to 'key' accepted truths that it uses" (Stylianides, 2009, p. 265). Again, it is not necessary for our purposes to unpack all the terms in the definition of proof (the reader can refer to Stylianides (2009) for elaboration). We clarify, though, that the term

* Corresponding author.
E-mail address: gabriel.stylianides@education.ox.ac.uk (G.J. Stylianides).

http://dx.doi.org/10.1016/j.jmathb.2016.04.002
0732-3123/

'accepted truths' is used broadly to include the axioms, theorems, definitions, and statements that a particular community may take as shared at a given time. Which accepted truths are 'key' and, thus, should be explicitly referenced in a proof depends on the audience of the proof (for example, some accepted truths may be considered trivial or basic knowledge for a particular audience and thus may be omitted from a proof). It is also important to note that our definition of 'proving' implies that this activity can include a cluster of other related activities that are often precursors to producing a proof such as testing examples to generalize and formulate conjectures, testing the conjectures against new evidence and revising the conjectures to conform with the evidence, and providing informal arguments that show the viability of the conjectures.

In this paper we adopt the subjective perspective we described above and we investigate conditions for proving by mathematical induction to be explanatory for the prover (or provers in the case of collaborative activity) as opposed to be explanatory for the reader (or readers) of the proof. Our adoption of the subjective perspective does not suggest that we consider it to be better than the absolutist perspective. We adopted the subjective perspective because of the particular focus of our study: we are interested in identifying conditions for proving by mathematical induction to be explanatory for university students working in small groups on proof problems. We say 'proving' as opposed to 'proof' because, contrary to most prior research on the functions of proof, we are not focusing on the final product of the proving activity (i.e., the proof) to examine whether that was explanatory for the provers. Rather, we focus on the proving activity that leads to the use of a particular method to develop a proof for a mathematical statement, with particular attention to provers' use of mathematical induction. In the next section we elaborate on the scope of the paper, beginning with a discussion of our notion of 'proving activity that is explanatory for provers' (or 'explanatory proving' for short).

2. Elaboration on the scope

To define our notion of explanatory proving, which is new in this paper, we will seek to maintain consistency with the way prior research defined *proof to be explanatory for the prover (or provers)*, namely, whether the proof illuminated or provided insight to a prover into why a mathematical statement is true (Bell, 1976; de Villiers, 1999; Hanna, 1990; Steiner, 1978) or false (Stylianides, 2009). We will thus consider the *proving activity to be explanatory for the prover (or provers)* if the method used in a proof provided a way for the prover to formalize the thinking that preceded and that illuminated or provided insight to the prover into why a statement is true or false.

This definition of explanatory proving is not specific to mathematical induction, but could apply to any proof method. Given our focus here on mathematical induction, however, we offer an example of how proving by mathematical induction could be explanatory for provers: provers could use *recursive reasoning* (that is, reasoning relating to or involving the repeated application of a rule or procedure to successive results) in their exploration of a mathematical statement in ways that could help provers see informally the structure of the inductive step in a possible proof by induction; the provers could subsequently apply mathematical induction to formalize their thinking and verify the truth of the statement.

Implied in our definition of explanatory proving is the idea that the proof followed naturally from the provers' investigation of a mathematical statement. This *continuity* between the final product (the proof) and the earlier parts of the proving process (the exploratory phase of proving) has similarities to the notion of *cognitive unity*. This notion derived from a long-term teaching experiment in Italy (e.g., Boero, Garuti, Lemut et al., 1996; Boero, Garuti, & Mariotti, 1996; Garuti, Boero, & Lemut, 1998) that aimed to introduce school students to Geometry and that focused on engaging students with problems that required both the development of a conjecture and its proof. The notion of cognitive unity was described and used in a number of different ways over the years (see Mariotti (2006, pp. 182–184) for a discussion). Below is one of the earliest and most commonly cited descriptions (Boero, Garuti, Lemut et al., 1996, p. 113); the original was in italics:

- during the production of the conjecture, the student progressively works out his/her statement through an intense argumentative activity functionally intermingling with the justification of the plausibility of his/her choices;
- during the subsequent statement proving stage, the student links up with this process in a coherent way, organising some of the justifications ("arguments") produced during the construction of the statement according to a logical chain.

As indicated in the previous excerpt, the notion of cognitive unity captures a possible continuity between the arguments that students produced to support or reject a specific conjecture and the final proof for the conjecture (Mariotti, 2006). The continuity in cognitive unity is similar to the continuity we described earlier in our definition of explanatory proving since, in both cases, there is a connection between the proof and parts of the proving process that preceded the proof's development. Yet the continuity in cognitive unity is more restricted than the one in our definition of explanatory proving. In cognitive unity the continuity is focused on the arguments that led to a conjecture and its proof. This is only one example of the kind of continuity we could have in our notion of explanatory proving. In explanatory proving, the provers could develop an insight (cf. the notion of 'conceptual insight' in Section 4.2) into the truth of a conjecture not necessarily because of an argument they developed, but because of recognizing a 'familiar territory' based on a particular representation (or 'appropriation of the statement' in Garuti et al.'s (1998) terms) that led them to see the relevance or usefulness of a particular proof method in turning their insight into an acceptable proof (cf. the notion of 'technical handle' in Section 4.2).

We decided to focus in this paper on mathematical induction not only because of its importance in the discipline of mathematics, but also because it is a proof method that is known to provide particular difficulties for students in achieving cognitive unity (Mariotti, 2006; pp. 187–188) and is frequently viewed by students as a proof that verifies without necessarily