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# Students' understanding of algebraic notation: A semiotic systems perspective

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ABSTRACT

The ideas of equivalence and variable are two of the most fundamental concepts in algebra. Most studies of students' understanding of these concepts have posited a gap between the students' conceptions and the institutional meanings for the symbols. In contrast, this study develops a theoretical framework for describing the ways undergraduate students use personal meanings for symbols as they appropriate institutional meanings. To do this, we introduce the idea of semiotic systems as a framework for understanding the ways students use collections of signs to engage in mathematical activity and how the students use these signs in meaningful ways. The analysis of students' work during task-based interviews suggests that this framework allows us to identify the ways in which seemingly idiosyncratic uses of the symbols are evidence of meaning-making and, in many cases, how the symbol use enables the student to engage productively in the mathematical activity.



## 1. Background

In addition to being a core topic in high school and college-level mathematics, algebra is a focal point of mathematics reform at all levels of schooling (e.g., Lacampagne, Blair, & Kaput, 1995; National Council of Teachers of Mathematics, 2000; National Research Council, 1998; RAND Mathematics Study Panel, 2003). Although much of the focus of these efforts has been on helping students develop algebraic reasoning, students must still develop fluency with algebraic symbols in order to fully engage with the concepts and to prepare for further study in mathematics.

The ideas of equivalence and variable are two of the most fundamental concepts in algebra (Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005). When students are initially learning algebra, their ability to use and describe the meaning of these concepts has been correlated with their success at solving algebra problems (Knuth, Stephens, McNeil, & Alibali, 2006). In particular, these concepts play a central role in core algebraic tasks, such as understanding relationships between quantities and describing patterns. Mathematics instructors typically ask students to think about these concepts by using their traditional symbolic representations: The equals sign and literal symbols such as $x$ and $y$.

Although there is a large body of research that describes the ways students think about these concepts and representations, we argue that this research provides an incomplete picture of how students understand and use algebraic symbols. In order to further enhance our understanding of students' algebraic thinking, we will present the idea of personal and institutional semiotic systems as a complement to more traditional cognitive perspectives.

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In the following sections, we describe some of the results of "traditional" research: Prior research on equivalence has described students as possessing two types of conceptions, prior research on variable has described students as gradually developing and possessing various conceptions, and prior research on multiplicative comparison has suggested that students frequently make a "reversal error." We describe how this research portrays students' conceptions as relatively stable, but recent research suggests that there is a more complex relationship between thinking and symbolizing. Then, we describe some alternative perspectives that have been developed to address this issue; these perspectives motivate the use of semiotics to understand students' mathematical activity.

### *1.1. Equivalence*

There are several meanings that mathematicians, teachers, and students tend to use for equivalence. Prediger (2010) summarized these various meanings: *operational,* where the equals sign indicates that a computation should be performed; *relational*, which indicates that two quantities or expressions are equivalent and focuses on the symmetric aspect of the equals sign; and *specification*, where a quantity is defined (e.g., $m := 1/2\ (a + b)$).

Numerous studies have described the ways students interpret the equals sign in elementary and middle grades and, to a lesser extent, in secondary grades. These studies have typically sought to characterize individual students as possessing a particular mental conception and describe how the conception facilitated or limited the students' ability to solve particular types of problems. For example, at the elementary level, Matthews, Rittle-Johnson, McEldoon, and Taylor (2012) classified students' conceptions of the equals sign in terms of operational and relational aspects and found that students who thought of the equals sign relationally tended to perform better when solving symbolic problems involving variables. At the middle-school level, researchers have investigated relationships between the students' interpretation of the equals sign and their strategies for solving various equivalence problems (e.g., Alibali, Knuth, Hattikudur, McNeil, & Stephens, 2007; Knuth et al., 2006), finding that students with relational conceptions were more successful at identifying equivalent equations. At the secondary and post-secondary levels researchers have investigated students' interpretations of and solution strategies using the equals sign (e.g., Godfrey & Thomas, 2008; Steinberg, Sleeman, & Ktorza 1991).

### *1.2. Variable*

As Schoenfeld and Arcavi (1988) and Philipp (1992) noted, there are numerous definitions for – and uses of – algebraic variables. Usiskin (1988) described the roles that variables play in four different conceptions of algebra: pattern generalizers, unknown or constant values, parameters or domain values of functions, and arbitrary elements of abstract algebraic structures. Similarly, Trigueros and Ursini (1999, 2001, 2003) described variables as being used as unknowns, general numbers, or related variables (i.e., as part of a function).

Researchers have typically associated students' understanding of variables with their interpretations of literal symbols (such as $x$ and $n$). Consequently, most studies of students' conceptions of variable have focused on middle and secondary grades. These studies have described how students interpret literal symbols as either labels for objects, as specific numbers, or as an arbitrary element of a particular set of numbers (e.g., Christou, Vosniadou, & Vamvakoussi, 2007; Kieran, 1990; Knuth et al., 2005; Küchemann, 1978; MacGregor & Stacey, 1997). Other studies have described students' difficulties using variables to represent quantities (e.g., White & Mitchelmore, 1996) and investigated the impact of using alternative symbolic approaches or pedagogical methods to help students' understand the institutional meanings of the symbols (e.g., Bardini, Pierce, & Stacey, 2004; Graham & Thomas, 2000).

In general, this research has portrayed students as possessing a particular conception of variable that gradually develops over time: They initially interpret letters as labels or specific numbers, and only later acquire the ability to treat letters as representing unknowns or variable quantities. Most researchers have concluded that students' struggles with variables – and algebra in general – are associated with developmental constraints or a failure to understand the meaning of operations performed on abstract symbols (e.g., Filloy & Rojano, 1989; Herscovics & Linchevski, 1994; MacGregor, 2001). However, as described below, some researchers have begun to offer alternative perspectives.

### *1.3. Multiplicative comparison*

Researchers have also investigated students' conceptions of equivalence and variable using multiplicative comparison problems. These problems involve representing a relationship between two quantities in which the value of one quantity is a constant multiple of the other, such as in the relationship "For every cake it sells, a bakery sells twelve brownies." Although the algebraic symbols that are required to work in the problem context are not complicated, students typically struggle to represent these relationships symbolically (e.g., Clement, 1982; Fisher, 1988; Kaput & Sims-Knight, 1983; Rosnick & Clement, 1980). In particular, in previous studies roughly 40–60% of college-level students provided incorrect responses, with many making a *reversal error*, in which they switched the roles of the literal symbols (e.g., writing $c = 12b$ instead of $b = 12c$ to represent the cake-brownie relationship).

Several researchers have attempted to explain the source of the reversal error, attributing the students' difficulties to cognitive misrepresentations of the symbols or to misconceptions about variables and/or equivalence. Some have suggested that the students may be syntactically translating the words into symbols (e.g., Clement, Lochhead, & Monk, 1981). However,