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A curious case of superscript (−1): Prospective secondary mathematics teachers explain

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ABSTRACT

In mathematics the same symbol – superscript (−1) – is used to indicate an inverse of a function and a reciprocal of a rational number. Is there a reason for using the same symbol in both cases? We analyze the responses of prospective secondary school teachers to this question. The responses are presented in a form of a dialogue between a teacher and a student and are accompanied with participants' commentary on their choices of instructional approaches. The data show that the majority of participants treat the symbol $\square^{-1}$ as a homonym, that is, the symbol is assigned different and unrelated meanings depending on a context. We discuss how knowledge of advanced mathematics (or lack of it) can guide instructional interaction.

Mathematics is the art of giving the same name to different things.
Henri Poincaré

1. Introduction

There is an ongoing conversation in mathematics education research on teacher knowledge and its various facets (e.g., Rowland & Ruthven, 2011). One important focus within this discussion is secondary teachers' "advanced mathematical knowledge" (AMK), defined as knowledge acquired during tertiary education (Zazkis & Leikin, 2010). Is this knowledge essential, or even useful, in teaching? Research demonstrated that teachers' opinions on the matter differ considerably, ranging from "irrelevant" to "extremely important" (Zazkis & Leikin, 2010).

However, even teachers who claim that AMK is essential for their teaching have difficulty in providing particular examples or recalling teaching scenarios where their AMK was utilized. Our study provides an example where a teacher's knowledge of advanced mathematics, or lack of it, can shape an instructional interaction. In particular, we attend to the conceptual connection between reciprocals and inverse functions, and the related symbol that represents these notions.

Our study was triggered by a particular interaction presented and discussed in Zazkis and D. Zazkis (2011).

Student: 3 to the negative 1 is one third [writes $3^{-1} = \frac{1}{3}$], right?

Rachel: Right.

Student: So power 'minus 1' means reciprocal, right?

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Rachel: Right.

Student: But f-to-minus-1 [writes f^{-1}] means the inverse function, this does not mean 1/f, right?

Rachel: Right.

Student: So they ran out of symbols, or what?

In the reported case Rachel was an experienced teacher educator and 'Student' was a prospective elementary school teacher. The authors briefly described Rachel's response as well as her hesitation with respect to her chosen approach. However, the described query can be raised by a secondary school student in a context of introduction to inverse functions. We wondered how prospective secondary school teachers would handle a similar situation with respect to the curious appearance of superscript (-1). Exploring this became the goal of our study.

2. On reciprocals, inverse functions, and the symbol of superscript (−1)

While a number followed by exponent (-1), points to the reciprocal of that number (e.g., $3^{-1} = \frac{1}{3}$), research literature attended to the two issues – reciprocals and negative exponents – separately. Within the abundant research on rational numbers, the discussion of reciprocals usually appears only in context of understanding or explaining division by fraction (e.g., Olive, 1999; Tirosh, 2000). Negative exponents are explored in the context of operations with exponents and problems associated with carrying out these operations (e.g., Cangelosi, Madrod, Coope, Olson, & Harter, 2013). We are not aware of a study that focused explicitly on the choice of a symbol to designate the reciprocal.

According to Cajori (1993), the notation $a^{-1} = \frac{1}{a}$ was chosen by Newton:

> "Our modern notation involving fractional and negative exponents was formally introduced [. . .] by Newton in a letter of June 13, 1676, to Oldenburg, then secretary of the Royal Society of London, which explains the use of negative and fractional exponents in the statement, "Since Algebraists write a^2, a^3, a^4, etc., for aa, aaa, aaaa, etc., so I write $a^{1/2}$, $a^{3/2}$, $a^{5/3}$ for $\sqrt{a}$, $\sqrt{a^3}$, $\sqrt[3]{a^5}$; and I write a^{-1}, a^{-2}, a^{-3}, etc., for $\frac{1}{a}$, $\frac{1}{aa}$, $\frac{1}{aaa}$ etc." (p. 355).

We interpret the phrase "since algebraist write – so I write" as a hint to consistency in the chosen notation with the exponential notation and operations on exponents, initially defined for positive integers. In particular, in the case of negative integer exponents, the definition $a^{-1} = \frac{1}{a}$ extends the equality $a^{m-n} = \frac{a^m}{a^n}$ for $m < n$. We note that a related issue, the case of zero exponent, was studied by Levenson (2012). She observed that mathematics teachers participating in her study have not identified $a^0 = 1$ as a definition that extends the notion of exponent (initially introduced as repeated multiplication), but considered the statement as a property to be proved.

The same symbol, superscript (-1), is also used in the context of functions, where f^{-1}, or $f^{-1}(x)$, is conventionally interpreted as the inverse function of $f(x)$. The choice of this notation is also connected to consistency in the use of exponents. As suggested by Todhunter (1863):

> "Experience will prove that the notation here given is often convenient. And we may shew [sic.] that it is not altogether an arbitrary notation, but one that naturally presents itself. For let any function of x be denoted as f(x). . . [. . .]
>
> We may examine what meaning it will be necessary to ascribe to $f^{-1}(x)$ in order that the relation $f^m f^n(x) = f^{m+n}(x)$ may hold when m or n is −1. Suppose that $m = 1$and $n = -1$; thus the relation becomes
>
> $ff^{-1}(x) = f^0(x) = x.$
>
> So that $f^{-1}(x)$ must denote a quantity whose function f is x ." (p. 205–206)

The last peculiar phrase "So that $f^{-1}(x)$ must denote a quantity whose function f is x"[1] can be rephrased as function f applied to $f^{-1}(x)$ results in x, that is, in a familiar notation, $f\left(f^{-1}(x)\right) = x$.

Within the large amount of research on functions, only a limited number of studies have focused on inverse functions. This research attended to the development of a concept of inverse function (e.g., Vidakovic, 1996) or misconceptions associated with inverse functions (e.g., Even, 1992; Lucus, 2006). The symbol used for designating inverse functions is treated in this research as an accepted convention and is not questioned.

It may appear initially surprising that the same symbol is used to designate reciprocals and inverse functions. However, the connection is evident from the perspective of group theory, where superscript (-1) denotes an inverse element in a group structure. In particular, the reciprocal of a number denotes the inverse of that number with respect to multiplication, while f^{-1} denotes the inverse of a function f with respect to composition of functions. Our study aims to explore how prospective secondary mathematics teachers respond to a student's question regarding the curious appearance of superscript (-1) in the two cases and explain the situation.

[1] The same phrase describing f^{-1} is found in Cajori (1928/1993, p. 270) and is attributed to J.F.W. Herschel (1820), *Calculus of Finite Differences*.